MATRIX ANALYSIS

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Book: Matrix Analysis (Cox)

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Preface to Matrix Analysis

TABLE OF CONTENTS

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Preface

Bellman has called matrix theory 'the arithmetic of higher mathematics.' Under the influence of Bellman and Kalman, engineers and scientists have found in matrix theory a language for representing and analyzing multivariable systems. Our goal in these notes is to demonstrate the role of matrices in the modeling of physical systems and the power of matrix theory in the analysis and synthesis of such systems.

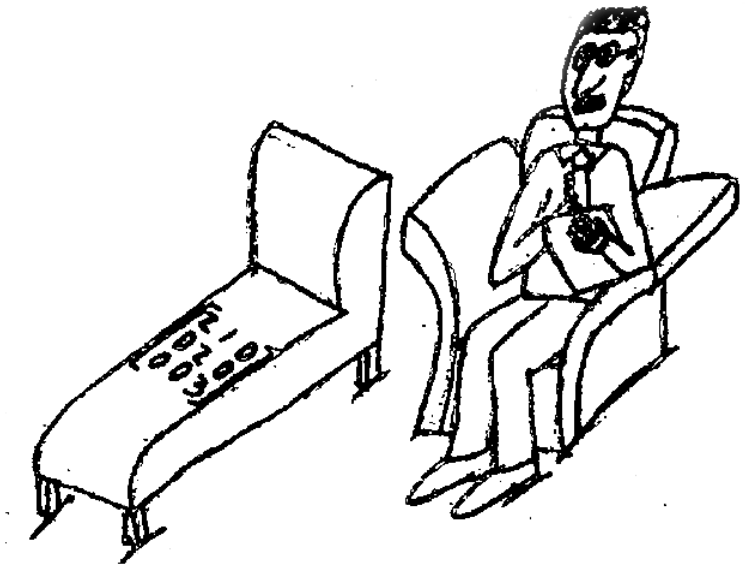

Figure 1: Matrix Analysis

Beginning with modeling of structures in static equilibrium we focus on the linear nature of the relationship between relevant state variables and express these relationships as simple matrix-vector products. For example, the voltage drops across the resistors in a network are linear combinations of the potentials at each end of each resistor. Similarly, the current through each resistor is assumed to be a linear function of the voltage drop across it. And, finally, at equilibrium, a linear combination (in minus out) of the currents must vanish at every node in the network. In short, the vector of currents is a linear transformation of the vector of voltage drops which is itself a linear transformation of the vector of potentials. A linear transformation of n numbers into m numbers is accomplished by multiplying the vector of n numbers by an m-by- n matrix. Once we have learned to spot the ubiquitous matrix-vector product we move on to the analysis of the resulting linear systems of equations. We accomplish this by stretching your knowledge of three-dimensional space. That is, we ask what does it mean that the m-by- n matrix X transforms Rn (real n-dimensional space) into Rm? We shall **visualize** this transformation by splitting both Rn and Rm each into two smaller spaces between which the given X behaves in very manageable ways. An understanding of this splitting of the ambient spaces into the so called **four fundamental subspaces** of X permits one to answer virtually every question that may arise in the study of structures in static equilibrium.

In the second half of the notes we argue that matrix methods are equally effective in the modeling and analysis of dynamical systems. Although our modeling methodology adapts easily to dynamical problems we shall see, with respect to analysis, that rather than splitting the ambient spaces we shall be better served by splitting X itself. The process is analogous to decomposing a complicated signal into a sum of simple harmonics oscillating at the natural frequencies of the structure under investigation. For we shall see that (most) matrices may be written as weighted sums of matrices of very special type. The weights are eigenvalues, or natural frequencies, of the matrix while the component matrices are projections composed from simple products of eigenvectors. Our approach to the eigendecomposition of matrices requires a brief exposure to the beautiful field of Complex Variables. This foray has the added benefit of permitting us a more careful study of the Laplace Transform, another fundamental tool in the study of dynamical systems.

--Steve Cox

CHAPTER OVERVIEW

1: Matrix Methods for Electrical Systems

1.1: Nerve Fibers and the Strang Quartet

We wish to confirm, by example, the prefatory claim that matrix algebra is a useful means of organizing (stating and solving) multivariable problems. In our first such example we investigate the response of a nerve fiber to a constant current stimulus. Ideally, a nerve fiber is simply a cylinder of radius aa and length l that conducts electricity both along its length and across its lateral membrane. Though we shall, in subsequent chapters, delve more deeply into the biophysics, here, in our first outing, we shall stick to its purely resistive properties. The latter are expressed via two quantities:

1. ρ_i the resistivity in Ωcm of the cytoplasm that fills the cell, and
2. ρ_m the resistivity in Ωcm^2 of the cell's lateral membrane.

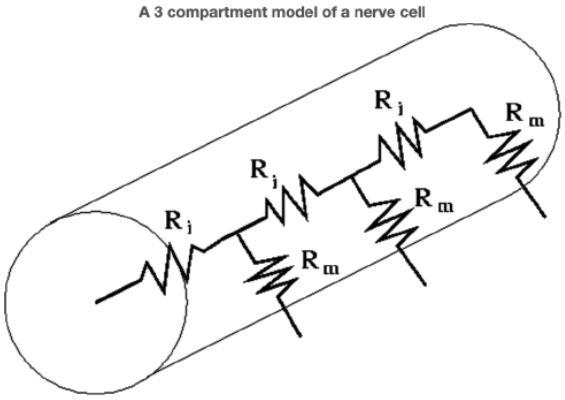

Figure 1.

Although current surely varies from point to point along the fiber it is hoped that these variations are regular enough to be captured by a multicompartment model. By that we mean that we choose a number N and divide the fiber into N segments each of length $\frac{l}{N}$ Denoting a segment's **axial resistance**

$$R_i = \frac{\rho_i \frac{l}{N}}{\pi a^2}$$

and **membrane resistance**

$$R_m = \frac{\rho_m}{2\pi a \frac{l}{N}}$$

we arrive at the lumped circuit model of Figure 1. For a fiber in culture we may assume a constant extracellular potential, e.g., zero. We accomplish this by connecting and grounding the extracellular nodes, see Figure 2.

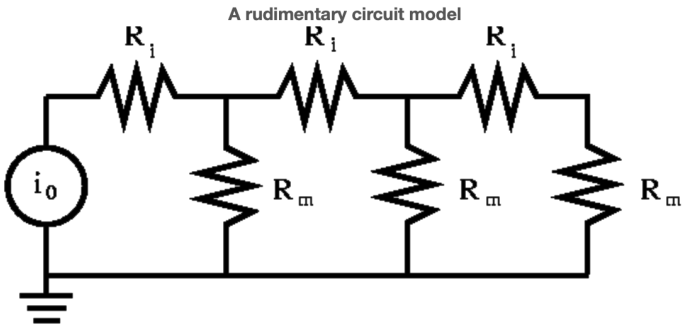

Figure 2.

Figure 2 also incorporates the **exogenous disturbance**, a current stimulus between ground and the left end of the fiber. Our immediate goal is to compute the resulting currents through each resistor and the potential at each of the nodes. Our long--range goal is to provide a modeling methodology that can be used across the engineering and science disciplines. As an aid to computing the desired quantities we give them names. With respect to figure 3, we label the vector of potentials

$$x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{pmatrix}$$

We have also (arbitrarily) assigned directions to the currents as a graphical aid in the consistent application of the basic circuit laws.

We incorporate the circuit laws in a modeling methodology that takes the form of a Strang Quartet:

- (S1) Express the voltage drops via $\mathbf{e} = -(A\mathbf{x})$
- (S2) Express **Ohm's Law** via $\mathbf{y} = G\mathbf{e}$
- (S3) Express **Kirchhoff's Current Law** via $A^T\mathbf{y} = -\mathbf{f}$
- (S4) Combine the above into $A^TGA\mathbf{x} = \mathbf{f}$

The A in (S1) is the **node-edge adjacency matrix** -- it encodes the network's connectivity. The G in (S2) is the diagonal matrix of edge conductances -- it encodes the physics of the network. The $\mathbf{f}$ in (S3) is the vector of current sources -- it encodes the network's stimuli. The culminating A^TGA in (S4) is the symmetric matrix whose inverse, when applied to $\mathbf{f}$, reveals the vector of potentials, $\mathbf{x}$. In order to make these ideas our own we must work many, many examples.

Example 1

Strang Quartet, Step 1

With respect to the circuit of Figure 3, in accordance with step 1, we express the six potential differences (always tail minus head)

$$e_1 = x_1 - x_2$$

$$e_2 = x_2$$

$$e_3 = x_2 - x_3$$

$$e_4 = x_3$$

$$e_5 = x_3 - x_4$$

$$e_6 = x_4$$

Such long, tedious lists cry out for matrix representation, to wit $\mathbf{e} = -(A\mathbf{x})$ where

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Strang Quartet, Step 2

Step 2, Ohm's Law, states:

The current along an edge is equal to the potential drop across the edge divided by the resistance of the edge.

In our case,

$$y_j = \frac{e_j}{R_i}, \quad j = 1, 3, 5 \quad and \quad y_j = \frac{e_j}{R_m}, \quad j = 2, 4, 6$$

or, in matrix notation, $\mathbf{y} = G\mathbf{e}$ where

$$A = \begin{pmatrix} \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_m} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R_m} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{R_m} \end{pmatrix}$$

Strang Quartet, Step 3

Step 3, Kirchhoff's Current Law, states:

The sum of the currents into each node must be zero.

In our case

$$i_0 - y_1 = 0$$

$$y_1 - y_2 - y_3 = 0$$

$$y_3 - y_4 - y_5 = 0$$

$$y_5 - y_6 = 0$$

or, in matrix terms

$$B\mathbf{y} = -\mathbf{f}$$

where

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad and \quad f = \begin{pmatrix} i_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Strang Quartet, Step 4

Looking back at A

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we recognize in B the **transpose** of A

- (S1) $\mathbf{e} = -(A\mathbf{x})$
- (S2) $\mathbf{y} = G\mathbf{e}$
- (S3) $A^T\mathbf{y} = -\mathbf{f}$

On substitution of the first two into the third we arrive, in accordance with (S4), at

$A^T GA\mathbf{x} = \mathbf{f}$

This is a system of four equations for the 4 unknown potentials, x_1 through x_4 As you know, the system Equation may have either 1, 0, or infinitely many solutions, depending on $\mathbf{f}$ and $A^T GA$ We shall devote (FIX ME CNXN TO CHAPTER 3 AND 4) to an unraveling of the previous sentence. For now, we cross our fingers and 'solve' by invoking the Matlab program

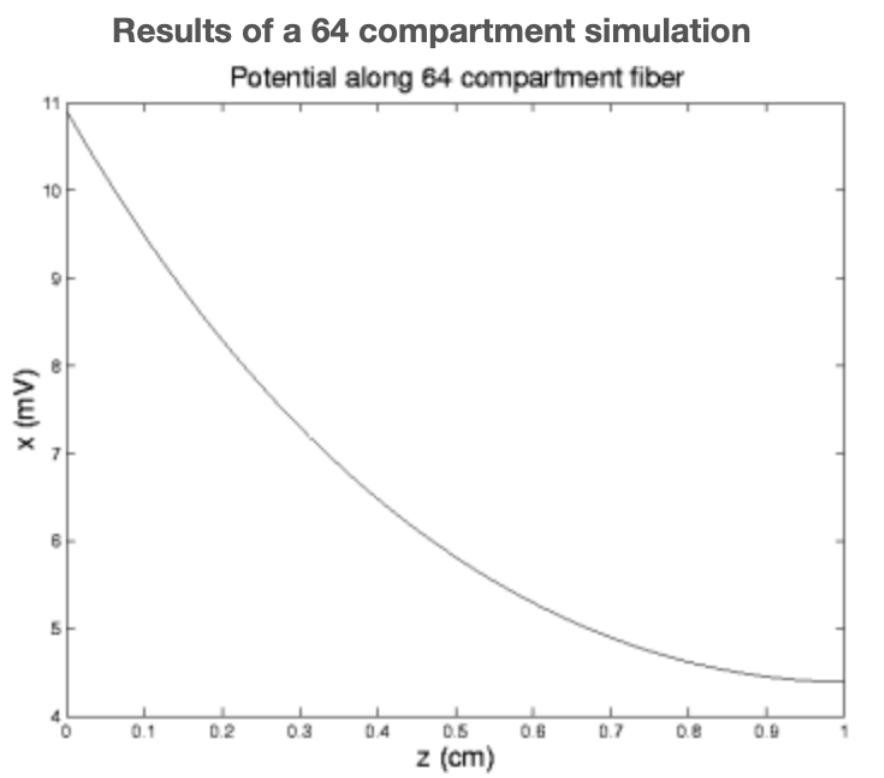

Figure 4.

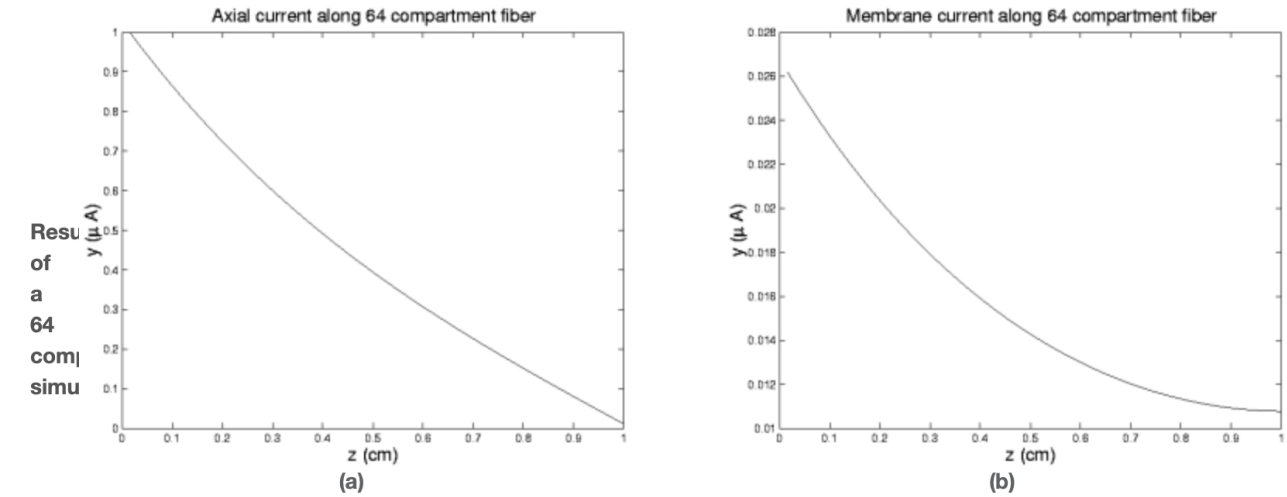

Figure 5.

This program is a bit more ambitious than the above in that it allows us to specify the number of compartments and that rather than just spewing the x and y values it plots them as a function of distance along the fiber. We note that, as expected, everything tapers off with distance from the source and that the axial current is significantly greater than the membrane, or leakage, current.

Example 2

We have seen in the previous example how a current source may produce a potential difference across a cell's membrane. We note that, even in the absence of electrical stimuli, there is always a difference in potential between the inside and outside of a living cell. In fact, this difference is the biologist's definition of 'living.' Life is maintained by the fact that the cell's interior is rich in potassium ions, K^+ and poor in sodium ions, Na^+ while in the exterior medium it is just the opposite. These concentration differences beget potential differences under the guise of the Nernst potentials:

Nernst potentials

$$E_{Na} = \frac{RT}{F} \log \frac{[Na]_o}{[Na]_i} \quad and \quad E_K = \frac{RT}{F} \log \frac{[K]_o}{[K]_i}$$

where R is the gas constant, T is temperature, and F is the Faraday constant.Associated with these potentials are membrane resistances

$$\rho_{m,Na} \quad and \quad \rho_{m,K}$$

that together produce the ρ_m above via

$$\frac{1}{\rho_m} = \frac{1}{\rho_{m,Na}} + \frac{1}{\rho_{m,K}}$$

and produce the aforementioned rest potential

$$E_m = \rho_m \left(\frac{E_{Na}}{\rho_{m,Na}} + \frac{E_K}{\rho_{m,K}} \right.$$

With respect to our old circuit model, each compartment now sports a battery in series with its membrane resistance, as shown in Figure 6.

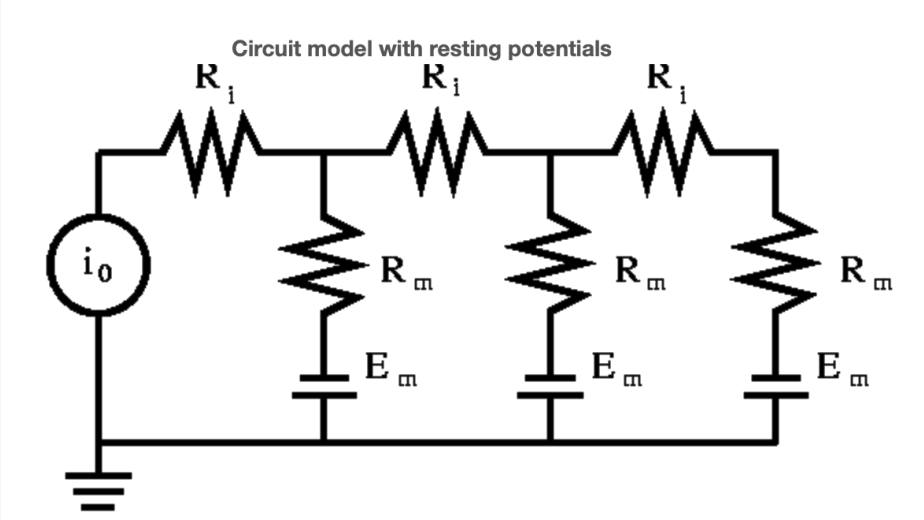

Figure 6.

Revisiting step (S1-4) we note that in (S1) the even numbered voltage drops are now

$$e_2 = x_2 - E_m$$

$$e_4 = x_3 - E_m$$

$$e_6 = x_4 - E_m$$

We accommodate such things by generalizing (S1) to:

- (S1') Express the voltage drops as $e = \mathbf{b} - A\mathbf{x}$ where $\mathbf{b}$ is the vector of batteries.

No changes are necessary for (S2) and (S3). The final step now reads,

- (S4') Combine (S1'), (S2), and (S3) to produce $A^T G A\mathbf{x} = A^T G\mathbf{bf}$

Returning to Figure 6, we note that

$$\mathbf{b} = -\begin{pmatrix} E_m \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} \quad and \quad A^T G\mathbf{b} = \frac{E_m}{R_m}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

This requires only minor changes to our old code. The results of its use are indicated in the next two figures.

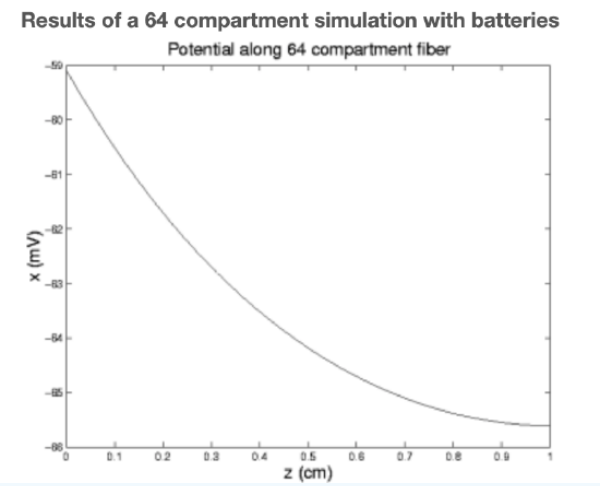

Figure 8.

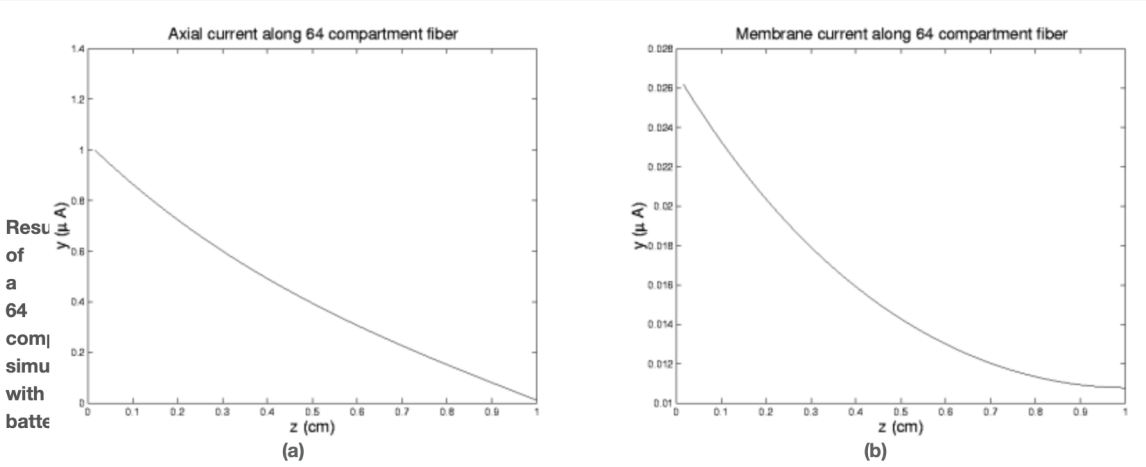

Figure 9.

1.2: Chapter 1 Exercises

Question 1.2.1

In order to refresh your matrix-vector multiply skills please calculate, by hand, the product A^TGA in the 3 compartment case and write out the 4 equations in the vector equation we arrived at in step(S4): $A^TGA\mathbf{x} = \mathbf{f}$

Feedback

The second equation should read

$$\frac{-x_1 + 2x_2 - x_3}{R_i} + \frac{x_2}{R_m} = 0 \tag{1.2.1}$$

Question 1.2.2

We began our discussion with the 'hope' that a multicompartment model could indeed adequately capture the fiber's true potential and current profiles. In order to check this one should run fib1.m with increasing values of NN until one can no longer detect changes in the computed potentials.

- (a) Please run fib1.m with $N = 8, 16, 32$, and 64. Plot all of the potentials on the **same** (use `hold`) graph, using different line types for each. (You may wish to alter `fib1.m` so that it accepts NN as an argument).

Let us now interpret this convergence. The main observation is that the difference Equation 1.2.1, approaches a differential equation. We can see this by noting that

$$\mathrm{d}(z) = \frac{l}{N} \tag{1.2.2}$$

acts as a spatial 'step' size and that $x_k\mathrm{d}(z)$ is approximately the value of the true potential at $(k-1)\mathrm{d}(z)$. In a slight abuse of notation, we denote the latter

$$x((k-1)\mathrm{d}(z)) \tag{1.2.3}$$

Applying these conventions to Equation 1.2.1 and recalling the definitions of R_i and R_m we see Equation 1.2.1 become

$$\frac{\pi a^2}{\rho_i}\frac{-x(0) + 2x(\mathrm{d}(z)) - x(2\mathrm{d}(z))}{\mathrm{d}(z)} + \frac{2\pi a\mathrm{d}(z)}{\rho_m}x(d(z)) = 0 \tag{1.2.4}$$

or, after multiplying through by $\frac{\rho_m}{\pi a\mathrm{d}(z)}$

$$\frac{a\rho_m}{\rho_i}\frac{-x(0) + 2x(\mathrm{d}(z)) - x(2\mathrm{d}(z))}{\mathrm{d}(z^2)} + 2x(\mathrm{d}(z)) = 0 \tag{1.2.5}$$

We note that a similar equation holds at each node (save the ends) and that as $N \to \infty$ and therefore $\mathrm{d}(z) \to 0$ we arrive at

$$\frac{d^2}{dz^2}x(z) - \frac{2\rho_i}{a\rho_m}x(z) = 0 \tag{1.2.6}$$

- (b) With $\mu \equiv \frac{2\rho_i}{a\rho_m}$ show that

$$x(z) = \alpha\sinh(\sqrt{2\mu}z) + \beta\cosh(\sqrt{2\mu}z) \tag{1.2.7}$$

satisfies Equation 1.2.5 regardless of α and β

We shall determine α and β by paying attention to the ends of the fiber. At the near end we find

$$\frac{\pi a^2}{\rho_i}\frac{x(0) - x(\mathrm{d}(z))}{\mathrm{d}(z)} = i_0 \tag{1.2.8}$$

which, as $\mathrm{d}(z) \to 0$ becomes

$$\frac{d}{dz}x(0) = -\frac{\rho_i i_0}{\pi a^2} \tag{1.2.9}$$

At the far end, we interpret the condition that no axial current may leave the last node to mean

$$\frac{d}{dz}x(l) = 0 \tag{1.2.10}$$

- (c) Substitute Equation 1.2.6 into Equation 1.2.7 and solve for α and β and write out the final $x(z)$.
- (d) Substitute into x the l, a, ρ_i, ρ_m values used in fib1.m, plot the resulting function (using, e.g., `ezplot`) and compare this to the plot achieved in part (a).

CHAPTER OVERVIEW

2: Matrix Methods for Mechanical Systems

2.1: A Uniaxial Truss

Introduction

We now investigate the mechanical prospection of tissue, an application extending techniques developed in the electrical analysis of a nerve cell. In this application, one applies traction to the edges of a square sample of planar tissue and seeks to identify, from measurement of the resulting deformation, regions of increased 'hardness' or 'stiffness.'

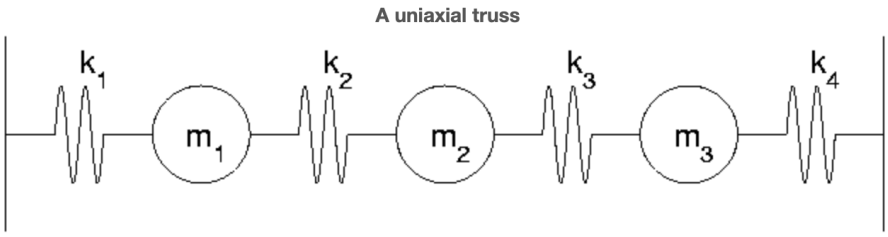

Figure 2.1.1: A Uniaxial Truss

As a precursor to the biaxial problem let us first consider the uniaxial case. We connect 3 masses with four springs between two immobile walls, apply forces at the masses, and measure the associated displacement. More precisely, we suppose that a horizontal force, f_j is applied to each m_j and produces a displacement x_j with the sign convention that rightward means positive. The bars at the ends of the figure indicate rigid supports incapable of movement. The k_j denote the respective spring stiffnesses. The analog of potential difference (see the electrical model) is here elongation. If e_j denotes the elongation of the jth spring then naturally,

$$e_1 = x_1$$

$$e_2 = x_2 - x_1$$

$$e_3 = x_3 - x_2$$

$$e_4 = -x_3$$

or, in matrix terms, $\mathbf{e} = A\mathbf{x}$ where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

We note that e_j is positive when the spring is stretched and negative when compressed. This observation, Hooke's Law, is the analog of Ohm's Law in the electrical model.

Hooke's Law

The restoring force in a spring is proportional to its elongation. We call the constant of proportionality the stiffness, k_j of the spring, and denote the restoring force by y_j. The mathematical expression of this statement is: $y_j = k_j e_j$ in matrix terms: $\mathbf{y} = K\mathbf{e}$ where

$$K = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}$$

The analog of Kirchhoff's Current Law is here typically called 'force balance.'

Force Balance

Equilibrium is synonymous with the fact that the net force acting on each mass must vanish. In symbols,

$$y_1 - y_2 - f_1 = 0$$

$$y_2 - y_3 - f_2 = 0$$

$$y_3 - y_4 - f_3 = 0$$

or, in matrix terms, $B\mathbf{y} = \mathbf{f}$ where

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \quad and \quad \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

As in the electrical example we recognize in B the transpose of A

$$\mathbf{e} = A\mathbf{x}$$

$$\mathbf{y} = K\mathbf{e}$$

$$A^T\mathbf{y} = \mathbf{f}$$

we arrive, via direct substitution, at an equation for $\mathbf{x}$. Namely,

$$(A^T\mathbf{y} = \mathbf{f}) \Rightarrow (A^TK\mathbf{e} = \mathbf{f}) \Rightarrow (A^TKA\mathbf{x} = \mathbf{f})$$

Assembling $A^TKA\mathbf{x}$ we arrive at the final system:

$$\begin{pmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

Gaussian Elimination and the Uniaxial Truss

Although Matlab solves systems like the one above with ease our aim here is to develop a deeper understanding of **Gaussian Elimination** and so we proceed by hand. This aim is motivated by a number of important considerations. First, not all linear systems have solutions and even those that do do not necessarily possess unique solutions. A careful look at Gaussian Elimination will provide the general framework for not only classifying those systems that possess unique solutions but also for providing detailed diagnoses of those defective systems that lack solutions or possess too many.

In Gaussian Elimination one first uses linear combinations of preceding rows to eliminate nonzeros below the main diagonal and then solves the resulting triangular system via back-substitution. To firm up our understanding let us take up the case where each $k_j = 1$ and so Equation takes the form

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

We eliminate the $(2,1)$ (row 2, column 1) element by implementing

$$\text{new row 2 = old row 2+}\tfrac{1}{2}\text{ row 1}$$

bringing

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 + \frac{f_1}{2} \\ f_3 \end{pmatrix}$$

We eliminate the current $(3,2)$ element by implementing

$$\text{new row 3=old row 3+}\tfrac{2}{3}\text{ row 2}$$

bringing the upper-triangular system

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 + \frac{f_1}{2} \\ f_3 + \frac{2f_1}{3} + \frac{f_1}{3} \end{pmatrix}$$

One now simply reads off

$$x_3 = \frac{f_1 + 2f_2 + 3f_3}{4}$$

This in turn permits the solution of the second equation

$$x_2 = \frac{2(x_3 + f_2 + \frac{f_1}{2})}{3} = \frac{f_1 + 2f_2 + f_3}{2}$$

and, in turn,

$$x_1 = \frac{x_2 + f_1}{2} = \frac{3f_1 + 2f_2 + f_3}{4}$$

One must say that Gaussian Elimination has succeeded here. For, regardless of the actual elements of $\mathbf{f}$, we have produced an $\mathbf{x}$ for which $A^T K A \mathbf{x} = \mathbf{f}$.

Alternate Paths to a Solution

Although Gaussian Elimination remains the most efficient means for solving systems of the form $S\mathbf{x} = \mathbf{f}$ it pays, at times, to consider alternate means. At the algebraic level, suppose that there exists a matrix that \undoes\ multiplication by SS in the sense that multiplication by 2^{-1} undoes multiplication by 2. The matrix analog of $2^{-1}2 = 1$ is

$$S^{-1}S = I$$

where I denotes the **identity matrix** (all zeros except the ones on the diagonal). We refer to S^{-1} as:

Inverse of S

Also dubbed "S inverse" for short, the value of this matrix stems from watching what happens when it is applied to each side of $S\mathbf{x} = \mathbf{f}$. Namely,

$$(S\mathbf{x} = \mathbf{f}) \Rightarrow (S^{-1}S\mathbf{x} = S^{-1}\mathbf{f}) \Rightarrow (I\mathbf{x} = S^{-1}\mathbf{f}) \Rightarrow (\mathbf{x} = S^{-1}\mathbf{f})$$

Hence, to solve $S\mathbf{x} = \mathbf{f}$ for $\mathbf{x}$ it suffices to multiply $\mathbf{f}$ by the inverse of S

Gauss-Jordan Method: Computing the Inverse of a Matrix

Let us now consider how one goes about computing S^{-1} In general this takes a little more than twice the work of Gaussian Elimination, for we interpret

$$S^{-1}S = I$$

as n (the size of S $\mathbf{f}$ running through nn columns of the identity matrix. The bundling of these nn applications into one is known as the **Gauss-Jordan method**. Let us demonstrate it on the S appearing in Equation. We first augment S with I

$$\begin{pmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

We then eliminate down, being careful to address each of the three $\mathbf{f}$ vectors. This produces

$$\begin{pmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix}$$

Now, rather than simple back--substitution we instead eliminate up. Eliminating first the $(2,3)$ element we find

$$\begin{pmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix}$$

In the final step we scale each row in order that the matrix on the left takes on the form of the identity. This requires that we multiply row 1 by $\frac{1}{2}$ row 2 by $\frac{3}{2}$ and row 3 by $\frac{3}{4}$ with the result

$$\begin{pmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

Now in this transformation of S into I we have, ipso facto, transformed I to S^{-1} i.e., the matrix that appears on the right after applying the method of Gauss-Jordan is the inverse of the matrix that began on the left. In this case,

$$S^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

One should check that $S^{-1}\mathbf{f}$ indeed coincides with the $\mathbf{x}$ computed above.

Invertibility

Not all matrices possess inverses:

singular matrix

A matrix that **does not** have an inverse.

Example

A simple example is:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Alternately, there are

Invertible, or Nonsingular Matrices

Matrices that **do** have an inverse.

Example

The matrix S that we just studied is invertible. Another simple example is

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

2.2: A Small Planar Truss

We return once again to the biaxial testing problem, introduced in the uniaxial truss module. It turns out that singular matrices are typical in the biaxial testing problem. As our initial step into the world of such planar structures let us consider the simple truss in the figure of a simple swing.

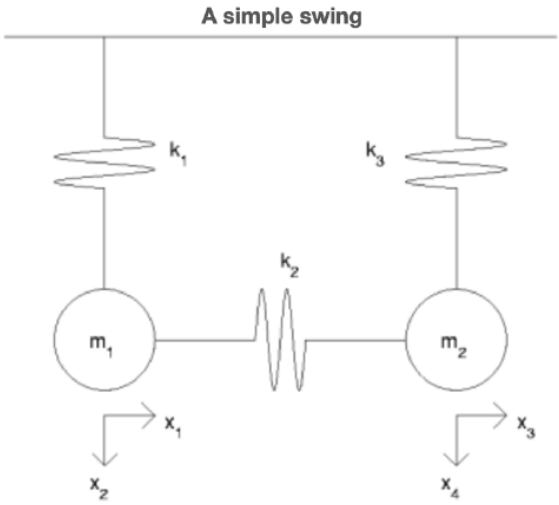

Figure 1.

We denote by x_1 and x_2 the respective horizontal and vertical displacements of m_1 (positive is right and down). Similarly, f_1 and f_2 will denote the associated components of force. The corresponding displacements and forces at m_2 will be denoted by x_3, x_4 and f_3, f_4 In computing the elongations of the three springs we shall make reference to their unstretched lengths, L_1, L_2 and L_3

Now, if spring 1 connects $\{0, -L_1\}$ to $\{0, 1\}$ when at rest and $\{0, -L_1\}$ to $\{x_1, x_2\}$ when stretched then its elongation is simply

$$e_1 = \sqrt{2x_1^2 + (x_2 + L_1)^2} - L_1$$

The price one pays for moving to higher dimensions is that lengths are now expressed in terms of square roots. The upshot is that the elongations are not linear combinations of the end displacements as they were in the uniaxial case. If we presume, however, that the loads and stiffnesses are matched in the sense that the displacements are small compared with the original lengths, then we may effectively ignore the nonlinear contribution in Equation. In order to make this precise we need only recall the Taylor development of the square root of $(1 + t)$

The Taylor development of $\sqrt{21 + t}$ about $t = 0$ is

$$\sqrt{21 + t} = 1 + \frac{t}{2} + O(t^2)$$

where the latter term signifies the remainder.

With regard to e_1 this allows

$$e_1 = \sqrt{2x_1^2 + x_2^2 + 2x_2 L_1 + L_1^2} - L_1$$

$$= L_1 \sqrt{21 + \frac{x_1^2 + x_2^2}{L_1^2} + \frac{2x_2}{L_1}} - L_1$$

$$e_1 = L_1 + \frac{x_1^2 + x_2^2}{2L_1^2} + x_2 + L_1 O((\frac{x_1^2 + x_2^2}{L_1^2} + \frac{2x_2}{L_1})^2) - L_1$$

$$= \frac{x_1^2 + x_2^2}{2L_1^2} + x_2 + L_1 O((\frac{x_1^2 + x_2^2}{L_1^2} + \frac{2x_2}{L_1})^2)$$

If we now assume that

$\frac{x_1^2+x_2^2}{L_1^2}$ is small compared with x_2

then, as the O term is even smaller, we may neglect all but the first terms in the above and so arrive at

$$e_1 = x_2$$

To take a concrete example, if L_1 is one meter and x_1 and x_2 are each one centimeter, then x_2 is one hundred times $\frac{x_1^2+x_2^2}{L_1^2}$.

With regard to the second spring, arguing as above, its elongation is (approximately) its stretch along its initial direction. As its initial direction is horizontal, its elongation is just the difference of the respective horizontal end displacements, namely,

$$e_2 = x_3 - x_1$$

Finally, the elongation of the third spring is (approximately) the difference of its respective vertical end displacements, i.e.,

$$e_3 = x_4$$

We encode these three elongations in

$$\mathbf{e} = A\mathbf{x} \quad where \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hooke's Law is an elemental piece of physics and is not perturbed by our leap from uniaxial to biaxial structures. The upshot is that the restoring force in each spring is still proportional to its elongation, i.e., $y_j = k_j e_j$ where k_j is the stiffness of the jth spring. In matrix terms,

$$\mathbf{y} = K\mathbf{e} \quad where \quad K = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}$$

Balancing horizontal and vertical forces at m_1 brings

$$(-y_2) - f_1 = 0$$

and

$$y_1 - f_2 = 0$$

while balancing horizontal and vertical forces at m_2 brings

$$y_2 - f_3 = 0$$

and

$$y_3 - f_4 = 0$$

We assemble these into

$$\mathbf{B}y = \mathbf{f} \quad where \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and recognize, as expected, that BB is nothing more than A^T. Putting the pieces together, we find that $\mathbf{x}$ must satisfy $S\mathbf{x} = \mathbf{f}$ where

$$S = A^T K A = \begin{pmatrix} k_2 & 0 & -k_2 & 0 \\ 0 & k_1 & 0 & 0 \\ -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_3 \end{pmatrix}$$

Applying one step of Gaussian Elimination brings

$$\begin{pmatrix} k_2 & 0 & -k_2 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_1 + f_3 \\ f_4 \end{pmatrix}$$

and back substitution delivers

$$x_4 = \frac{f_4}{k_3}$$

$$0 = f_1 + f_3$$

$$x_2 = \frac{f_2}{k_1}$$

$$x_1 - x_3 = \frac{f_1}{k_2}$$

The second of these is remarkable in that it contains no components of $\mathbf{x}$. Instead, it provides a condition on $\mathbf{f}$. In mechanical terms, it states that there can be no equilibrium unless the horizontal forces on the two masses are equal and opposite. Of course one could have observed this directly from the layout of the truss. In modern, three--dimensional structures with thousands of members meant to shelter or convey humans one should not however be satisfied with the `visual' integrity of the structure. In particular, one desires a detailed description of all loads that can, and, especially, all loads that can not, be equilibrated by the proposed truss. In algebraic terms, given a matrix S, one desires a characterization of

1. all those $\mathbf{f}$ for which $S\mathbf{x} = \mathbf{f}$ **possesses** a solution
2. all those $\mathbf{f}$ for which $S\mathbf{x} = \mathbf{f}$ **does not** possess a solution

We will eventually provide such a characterization in our later discussion of the column space of a matrix.

Supposing now that $f_1 + f_3 = 0$ we note that although the system above is consistent it still fails to uniquely determine the four components of $\mathbf{x}$. In particular, it specifies only the difference between x_1 and x_3 As a result both

$$\mathbf{x} = \begin{pmatrix} \frac{f_1}{k_2} \\ \frac{f_2}{k_1} \\ 0 \\ \frac{f_4}{k_3} \end{pmatrix} \quad and \quad \mathbf{x} = \begin{pmatrix} 0 \\ \frac{f_2}{k_1} \\ -\frac{f_1}{k_2} \\ \frac{f_4}{k_3} \end{pmatrix}$$

satisfy $S\mathbf{x} = \mathbf{f}$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and still have a solution of $S\mathbf{x} = \mathbf{f}$. Searching for the source of this lack of uniqueness we observe some redundancies in the columns of S. In particular, the third is simply the opposite of the first. As S is simply $A^T K A$, where again, the first and third columns are opposites. These redundancies are encoded in $\mathbf{z}$ in the sense that

$$A\mathbf{z} = \mathbf{0}$$

Interpreting this in mechanical terms, we view $\mathbf{z}$ as a displacement and $A\mathbf{z}$ as the resulting elongation. In

$$A\mathbf{z} = \mathbf{0}$$

we see a nonzero displacement producing zero elongation. One says in this case that the truss deforms without doing any work and speaks of $\mathbf{z}$ as an **unstable mode**. Again, this mode could have been observed by a simple glance at Figure. Such is not the case for more complex structures and so the engineer seeks a systematic means by which **all** unstable modes may be identified. We shall see later that all these modes are captured by the null space of A.

From

$$S\mathbf{z} = \mathbf{0}$$

one easily deduces that S is singular. More precisely, if S^{-1} were to exist then $S^{-1}S\mathbf{z}$ would equal $S^{-1}\mathbf{0}$ i.e. $\mathbf{z} = \mathbf{0}$, contrary to Equation. As a result, Matlab will fail to solve $S\mathbf{x} = \mathbf{f}$ even when $\mathbf{f}$ is a force that the truss can equilibrate. One way out is to use the **pseudo-inverse**, as we shall see in the General Planal Truss module.

2.3: The General Planar Truss

Let us now consider something that resembles the mechanical prospection problem introduced in the introduction to matrix methods to matrix methods for mechanical systems. In the figure below we offer a crude mechanical model of a planar tissue, say, e.g., an excised sample of the wall of a vein.

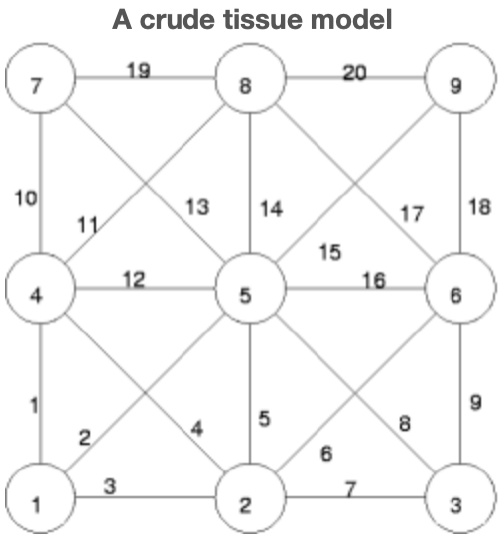

Figure 1.

Elastic fibers, numbered 1 through 20, meet at nodes, numbered 1 through 9. We limit our observation to the motion of the nodes by denoting the horizontal and vertical displacements of node j by x_{2j-1} (horizontal) and x_{2j} (vertical), respectively. Retaining the convention that down and right are positive we note that the elongation of fiber 1 is

$$e_1 = x_2 - x_8$$

while that of fiber 3 is

$$e_3 = x_3 - x_1$$

.

As fibers 2 and 4 are neither vertical nor horizontal their elongations, in terms of nodal displacements, are not so easy to read off. This is more a nuisance than an obstacle however, for noting our discussion of elongation in the small planar truss module, the elongation is approximately just the stretch along its undeformed axis. With respect to fiber 2, as it makes the angle $-\frac{\pi}{4}$ with respect to the positive horizontal axis, we find

$$e_2 = x_9 \cos(-\frac{\pi}{4}) - x_{10} \sin(-\frac{\pi}{4}) = \frac{x_9 - x_1 + x_2 - x_{10}}{\sqrt{22}}$$

Similarly, as fiber 4 makes the angle $-\frac{3\pi}{4}$ with respect to the positive horizontal axis, its elongation is

$$e_4 = x_7 \cos(-\frac{3\pi}{4}) - x_8 \sin(-\frac{3\pi}{4}) = \frac{x_3 - x_7 + x_4 - x_8}{\sqrt{22}}$$

These are both direct applications of the general formula

$$e_j = x_{2n-1} \cos(\theta_j) - x_{2n} \sin(\theta_j)$$

for fiber j Figure below, connecting node mm to node nn and making the angle θ_j with the positive horizontal axis when node m is assumed to lie at the point $(0, 0)$. The reader should check that our expressions for e_1 and e_3 indeed conform to this general formula and that e_2 and e_4 agree with ones intuition. For example, visual inspection of the specimen suggests that fiber 2 can not be supposed to stretch (i.e., have positive e_2) unless $x_9 > x_1$ and/or $x_2 > x_{10}$. Does this jive with Equation?

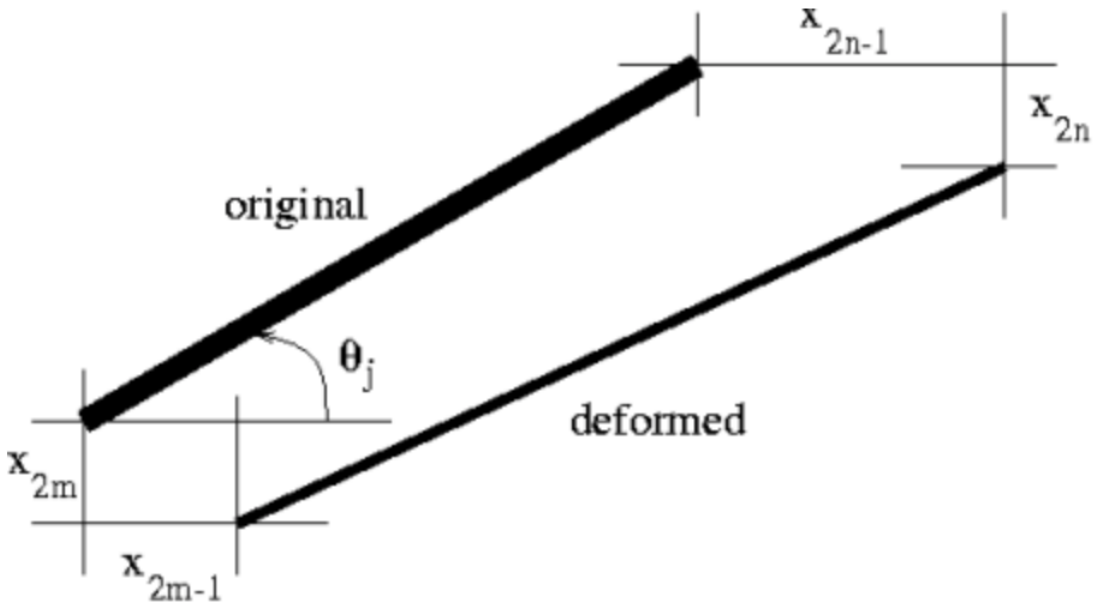

Figure 2. Elongation of a generic bar, see Equation.

Applying Equation to each of the remaining fibers we arrive at $\mathbf{e} = A\mathbf{x}$ where A is 20-by-18, one row for each fiber, and one column for each degree of freedom. For systems of such size with such a well defined structure one naturally hopes to automate the construction. We have done just that in the accompanying M-file and diary. The M-file begins with a matrix of raw data that anyone with a protractor could have keyed in directly from Figure 1.:

```
data =     [                                % one row of data for each fiber, the
1       4       -pi/2           % first two columns are starting and ending
1       5       -pi/4           % node numbers, respectively, while the third is the
1       2       0               % angle the fiber makes with the positive horizontal axis
2       4       -3*pi/4
...and so on...                  ]
```

This data is precisely what Euqation requires in order to know which columns of A receive the proper cos or sin. The final A matrix is displayed in the diary.

The next two steps are now familiar. If K denotes the diagonal matrix of fiber stiffnesses and $\mathbf{f}$ denotes the vector of nodal forces then

$$\mathbf{y} = K\mathbf{e} \quad and \quad A^T\mathbf{y} = \mathbf{f}$$

and so one must solve $S\mathbf{x} = \mathbf{f}$ where $S = A^T K A$. In this case there is an entire three--dimensional class of $\mathbf{z}$ for which $A\mathbf{z} = \mathbf{0}$ and therefore $S\mathbf{z} = \mathbf{0}$ e.g., two translations and a rotation. As a result S is singular and `x = S\f` in MATLAB will get us nowhere. The way out is to recognize that S has $18 - 3 = 15$ stable modes and that if we restrict S to 'act' only in these directions then it 'should' be invertible. We will begin to make these notions precise in discussions on the Fundamental Theorem of Linear Algebra. For now let us note that every matrix possesses such a **pseudo-inverse** and that it may be computed in MATLAB via the `pinv` command. Supposing the fiber stiffnesses to each be one and the edge traction to be of the form

$$\mathbf{f} = (-1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad -1 \quad -1 \quad 0 \quad -1 \quad 1 \quad -1)^T$$

,

we arrive at $\mathbf{x}$ via `x=pinv(S)*f` and offer below its graphical representation.

Before-After Plot

2.4: Chapter 2 Exercises

Exercise 2.4.1

With regard to the unixial truss figure,

i. Derive the A and K matrices resulting from the removal of the fourth spring,
ii. Compute the inverse, by hand via Gauss-Jordan, of the resulting A^TKA with $k_1 = k_2 = k_3 = k$
iii. Use the result of (ii) to find the displacement corresponding to the load $\mathbf{f} = (0, 0, F)^T$

Exercise 2.4.2

Generalize example 3, the general planar truss, to the case of 16 nodes connected by 42 fibers. Introduce one stiff (say $k = 100$) fiber and show how to detect it by 'properly' choosing $\mathbf{f}$ the before-after plot in the general planar module, from which you conclude the presence of a stiff fiber.

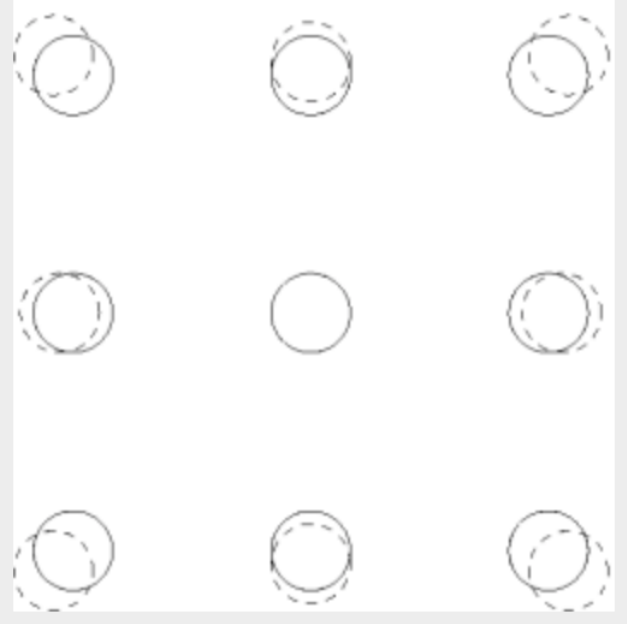

Figure 1. A copy of the before-after figure from the general planar module.

CHAPTER OVERVIEW

3: The Fundamental Subspaces

3.1: Column Space

We begin with the simple geometric interpretation of matrix-vector multiplication. Namely, the multiplication of the n-by-1 vector x by the m-by-n matrix A produces a linear combination of the columns of A. More precisely, if a_j denotes the jth column of A then

$$Ax = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

The picture that I wish to place in your mind's eye is that AxAx lies in the subspace spanned by the columns of A. This subspace occurs so frequently that we find it useful to distinguish it with a definition.

Column Space

The column space of the m-by-n matrix S is simply the span of the its columns, i.e. $Ra(S) \equiv \{Sx | x \in \mathbb{R}^n\}$ subspace of $\mathcal{R}^m$ stands for range in this context.The notation R_a stands for range in this context.

Example

Let us examine the matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The column space of this matrix is:

$$Ra(A) = \{x_1 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} | x \in \mathbb{R}^4\}$$

As the third column is simply a multiple of the first, we may write:

$$Ra(A) = \{x_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} | x \in \mathbb{R}^3\}$$

As the three remaining columns are linearly independent we may go no further. In this case, $Ra(A)$ comprises all of $\mathbb{R}^3$

Method for Finding a Basis

To determine the basis for $Ra(A)$ (where A is an arbitrary matrix) we must find a way to discard its dependent columns. In the example above, it was easy to see that columns 1 and 3 were colinear. We seek, of course, a more systematic means of uncovering these, and perhaps other less obvious, dependencies. Such dependencies are more easily discerned from the row reduced form. In the reduction of the above problem, we come very easily to the matrix

$$A_{red} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Once we have done this, we can recognize that the pivot column are **the** linearly independent columns of A_{red}. One now asks how this might help us distinguish the independent columns of A. For, although the rows of A_{red} are linear combinations of the rows of A **pay attention to the indices of the pivot columns**. In our example, columns $\{1, 2, 4\}$ are the pivot columns of A_{red} and hence the first, second, and fourth columns of A i.e.,

$$\left\{\begin{pmatrix}0\\-1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

comprise a basis for $Ra(A)$:

Definition: A Basis for the Column Space

Suppose A is m-by-n. If columns $\{c_j|j=1,\cdots,r\}$ are the pivot columns of A_{red} then columns $\{c_j|j=1,\cdots,r\}$ of A constitute a basis for $Ra(A)$

3.2: Null Space

Definition: Null Space

The null space of an m-by-n matrix A is the collection of those vectors in $\mathbb{R}^n$ that A maps to the zero vector in $\mathbb{R}^m$. More precisely,

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n | Ax = 0\}$$

Null Space Example

As an example, we examine the matrix A

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is fairly easy to see that the null space of this matrix is:

$$\mathcal{N}(A) = \{t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} | t \in \mathbb{R}\}$$

This is a line in $\mathbb{R}^4$

The null space answers the question of uniqueness of solutions to $S\mathbf{x} = \mathbf{f}$. For, if $S\mathbf{x} = \mathbf{f}$ and $S\mathbf{y} = \mathbf{f}$ then $S(\mathbf{x} - \mathbf{y}) = S\mathbf{x} - S\mathbf{y} = \mathbf{f} - \mathbf{f} = 0$ and so $(\mathbf{x} - \mathbf{y}) \in \mathcal{N}(S)$. Hence, a solution to $S\mathbf{x} = \mathbf{f}$ will be unique if, and only if, $\mathcal{N}S = \{0\}$

Method for Finding the Basis

Let us now exhibit a basis for the null space of an arbitrary matrix A. We note that to solve $A\mathbf{x} = 0$ is to solve $A_{red}\mathbf{x} = 0$. With respect to the latter, we suppose that

$$\{c_j | j = \{1, \cdots, r\}\}$$

are the indices of the pivot columns and that

$$\{c_j | j = \{r+1, \cdots, n\}\}$$

are the indices of the nonpivot columns. We accordingly define the r pivot variables

$$\{x_{c_j} | j = \{1, \cdots, r\}\}$$

and the $n - r$ free variables

$$\{x_{c_j} | j = \{r+1, \cdots, n\}\}$$

One solves $A_{red}\mathbf{x} = 0$ by expressing each of the pivot variables in terms of the nonpivot, or free, variables. In the example above, x_1, x_2, and x_4 are pivot while x_3 is free. Solving for the pivot in terms of the free, we find $x_4 = 0, x_3 = x_1, x_2 = 0$, or, written as a vector,

$$\mathbf{x} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

where x_3 is free. As x_3 ranges over all real numbers the x above traces out a line in $\mathbb{R}_4$. This line is precisely the null space of A. Abstracting these calculations we arrive at:

Definition: A Basis for the Null Space

Suppose that A is m-by-n with pivot indices $\{c_j | j = \{1, \cdots, r\}\}$ and free indices $\{c_j | j = \{r+1, \cdots, n\}\}$. A basis for $\mathcal{N}(A)$ may be constructed $n-r$ vectors $\{z^1, z^2, \cdots, z^{n-r}\}$ where z^k, and only z^k possesses a nonzero in its c_{r+k} component.

A MATLAB Observation

As usual, MATLAB has a way to make our lives simpler. If you have defined a matrix A and want to find a basis for its null space, simply call the function `null(A)`. One small note about this function: if one adds an extra flag, `'r'`, as in `null(A, 'r')`, then the basis is displayed "rationally" as opposed to purely mathematically. The MATLAB help pages define the difference between the two modes as the rational mode being useful pedagogically and the mathematical mode of more value **(gasp!)** mathematically.

Final thoughts on null spaces

There is a great deal more to finding null spaces; enough, in fact, to warrant another module. One important aspect and use of null spaces is their ability to inform us about the uniqueness of solutions. If we use the column space to determine the existence of a solution $\mathbf{x}$ to the equation $A\mathbf{x} = b$. Once we know that a solution exists it is a perfectly reasonable question to want to know whether or not this solution is the only solution to this problem. The hard and fast rule is that a solution $\mathbf{x}$ is unique if and only if the null space of A is empty. One way to think about this is to consider that if $A\mathbf{x} = 0$ does not have a unique solution then, by linearity, neither does $A\mathbf{x} = b$. Conversely, if $(Az = 0) \wedge (z \neq 0) \wedge (A\mathbf{y} = b)$ then $A(z + \mathbf{y}) = b$ as well.

3.3: The Null and Column Spaces- An Example

Preliminary Information

Let us compute bases for the null and column spaces of the adjacency matrix associated with the ladder below.

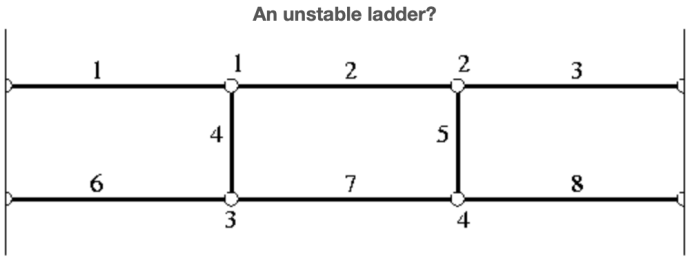

Figure 1.

The ladder has 8 bars and 4 nodes, so 8 degrees of freedom. Denoting the horizontal and vertical displacements of node j by x_{2j-1} and x_{2j} respectively, we arrive at the A matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Finding a Basis for the Column Space

To determine a basis for $\mathcal{R}(A)$ we must find a way to discard its dependent columns. A moment's reflection reveals that columns 2 and 6 are colinear, as are columns 4 and 8. We seek, of course, a more systematic means of uncovering these and perhaps other less obvious dependencies. Such dependencies are more easily discerned from the row reduced form

$$A_{red} = rref(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Recall that `rref` performs the elementary row operations necessary to eliminate all nonzeros below the diagonal. For those who can't stand to miss any of the action I recommend `rrefmovie` .

NOT_CONVERTED_YET: para

Each nonzero row of

A_{red} is called a **pivot row**. The first nonzero in each row of A_{red} is called a **pivot**. Each column that contains a pivot is called a **pivot column**. On account of the staircase nature of A_{red} we find that there are as many pivot columns as there are pivot rows. In our example there are six of each and, again on account of the staircase nature, the pivot columns are **the linearly independent** columns of A_{red} One now asks how this might help us distinguish the independent columns of A. For although the rows of A_{red} are linear combinations of the rows of A, no such thing is true with respect to the columns. In our example, columns $\{1, 2, 3, 4, 5, 7\}$are the pivot columns. In general:

Proposition

Suppose A is m-by-n. If columns $\{c_j|j=1,\cdots,r\}$ are the pivot columns of A_{red} then columns $\{c_j|j=1,\cdots,r\}$ of A constitute a basis for $\mathcal{R}(A)$.

Note that the pivot columns of AredAred are, by construction, linearly independent. Suppose, however, that columns $\{c_j|j=1,\cdots,r\}$ of A are linearly dependent. In this case there exists a nonzero $x\in\mathbb{R}^n$ for which $A\mathbf{x}=\mathbf{0}$ and

$$\forall k, k\notin\{c_j|j=1,\cdots,r\}:(x_k=0)$$

Now $A\mathbf{x}=\mathbf{0}$ necessarily implies that $A_{red}\mathbf{x}=\mathbf{0}$ contrary to the fact that columns $\{c_j|j=1,\cdots,r\}$ are the pivot columns of A_{red}

We now show that the span of columns $\{c_j|j=1,\cdots,r\}$ of A indeed coincides with $\mathbb{R}(A)$ This is obvious if $r=n$ i.e., if **all** of the columns are linearly independent. If $r<n$ there exists a $q\notin\{c_j|j=1,\cdots,r\}$ Looking back at A_{red} we note that its qth column is a linear combination of the pivot columns with indices not exceeding q. Hence, there exists an x satisfying Equation and $A_{red}\mathbf{x}=\mathbf{0}$ and $x_q=1$ This x then necessarily satisfies $A\mathbf{x}=\mathbf{0}$. This states that the qth column of A is a linear combination of columns $\{c_j|j=1,\cdots,r\}$ of A

Finding a Basis for the Null Space

Let us now exhibit a basis for $\mathcal{N}(A)$ We exploit the already mentioned fact that $\mathcal{N}(A)=\mathcal{N}(A_{red})$. Regarding the latter, we partition the elements of $\mathbf{x}$ into so called **pivot variables**,

$$\{x_{c_j}|j=1,\cdots,r\}$$

and **free variables**

$$\{x_k|k\notin\{x_j|j=1,\cdots,r\}\}$$

There are evidently $n-r$ free variables. For convenience, let us denote these in the future by

$$\{x_{c_j}|j=r+1,\cdots,n\}$$

One solves $A_{red}\mathbf{x}=\mathbf{0}$ by expressing each of the pivot variables in terms of the nonpivot, or free, variables. In the example above, x_1,x_2,x_3,x_4,x_5 and x_7 are pivot while x_6 and x_8 are free. Solving for the pivot in terms of the free we find

$$x_7=0 \tag{3.3.1}$$

$$x_5=0 \tag{3.3.2}$$

$$x_4=x_8 \tag{3.3.3}$$

$$x_3=0 \tag{3.3.4}$$

$$x_2=x_6 \tag{3.3.5}$$

$$x_1=0 \tag{3.3.6}$$

or, written as a vector,

$$\mathbf{x}=x_6\begin{pmatrix}0\\1\\0\\0\\0\\1\\0\\0\end{pmatrix}+x_8\begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\1\end{pmatrix}$$

where x_6 and x_8 are free. As x_6 and x_8 range over all real numbers, the $\mathbf{x}$ above traces out a plane in $\mathbb{R}^8$ This plane is precisely the null space of A and Equation describes a generic element as the linear combination of two basis vectors. Compare this to what

MATLAB returns when faced with `null(A,'r')` . Abstracting these calculations we arrive at

Proposition

Suppose that A is m-by-n with pivot indices $\{c_j | j = 1, \cdots, r\}$ and free indices $\{c_j | j = r+1, \cdots, n\}$ A basis for $\mathcal{N}(A)$ may be constructed of $n-r$ vectors $\{z^1, z^2, \cdots, z^{n-r}\}$ where z^k and only z^k possesses a nonzero in its c_{r+k} component.

The Physical Meaning of Our Calculations

Let us not end on an abstract note however. We ask what $\mathcal{R}(A)$ and $\mathcal{N}(A)$ tell us about the ladder. Regarding $\mathcal{R}(A)$ the answer will come in the next chapter. The null space calculation however has revealed two independent motions against which the ladder does no work! Do you see that the two vectors in Equation encode rigid vertical motions of bars 4 and 5 respectively? As each of these lies in the null space of A Figure? I hope not, for vertical motion of bar 4 must 'stretch' bars 1, 2, 6, and 7. How does one resolve this (apparent) contradiction?

3.4: Left Null Space

If one understands the concept of a null space, the left null space is extremely easy to understand.

Definition: Left Null Space

The Left Null Space of a matrix is the null space of its transpose, i.e.,

$$\mathcal{N}(A^T) = \{\mathbf{y} \in \mathbb{R}^m | A^T\mathbf{y} = 0\}$$

The word "left" in this context stems from the fact that $A^T\mathbf{y} = 0$ is equivalent to $\mathbf{y}^T A = 0$ where $\mathbf{y}$ "acts" on A from the left.

Example

As A_{red} was the key to identifying the null space of A, we shall see that A^T_{red} is the key to the null space of A^T. If

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

then

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

and so

$$A^T_{red} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

We solve $A^T_{red} = 0$ by recognizing that y_1 and y_2 are pivot variables while y_3 is free. Solving $A^T_{red}\mathbf{y} = 0$ for the pivot in terms of the free we find $y_2 = -(2y_3)$ and $y_1 = y_3$ hence

$$\mathcal{N}(A^T) = \left\{ y_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} | y_3 \in \mathbb{R} \right\}$$

Finding a Basis for the Left Null Space

The procedure is no different than that used to compute the null space of A itself. In fact

Definition: A Basis for the Left Null Space

Suppose that A^T is n-by-m with pivot indices $\{c_j | j = \{1, \cdots, r\}\}$ and free indices $\{c_j | j = \{r+1, \cdots, n\}\}$. A basis for $\mathcal{N}(A^T)$ may be constructed of $m - r$ vectors $\{z^1, z^2, \cdots, z^{m-r}\}$ where z^k and only z^k, possesses a nonzero in its c_{r+k} component.

3.5: Row Space

The Row Space

As the columns of A^T are simply the rows of A we call $Ra(A^T)$ the row space of A^T. More precisely

Definition: Row Space

The row space of the m-by-n matrix A is simply the span of its rows, i.e.,

$$Ra(A^T) \equiv \{A^T\mathbf{y} | \mathbf{y} \in \mathbb{R}^m\}$$

This is a subspace of $\mathbb{R}^n$

Let us examine the matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The row space of this matrix is:

$$\mathscr{R}\mathrm{a}(A^T) = \left\{ y_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + y_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} | y \in \mathbb{R}^3 \right\}$$

As these three rows are linearly independent we may go no further. We "recognize" then $\mathcal{R}a(A^T)$ as a three dimensional subspace of $\mathbb{R}^4$

Method for Finding the Basis of the Row Space

Regarding a basis for $\mathscr{R}\mathrm{a}(A^T)$ we recall that the rows of A_{red}, the row reduced form of the matrix A, are merely linear A combinations of the rows of A and hence

$$\mathscr{R}\mathrm{a}(A^T) = \mathscr{R}\mathrm{a}(A_{red})$$

This leads immediately to:

Definition: A Basis for the Row Space

Suppose A is m-by-n. The pivot rows of A_{red} constitute a basis for $\mathscr{R}\mathrm{a}(A^T)$.

With respect to our example,

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

comprises a basis for $\mathscr{R}\mathrm{a}(A^T)$.

3.6: Exercises- Columns and Null Spaces

Exercise 3.6.1

I encourage you to use `rref` and `null` for the following.

i. Add a diagonal crossbar between nodes 3 and 2 in the unstable ladder figure and compute bases for the column and null spaces of the new adjacency matrix. As this crossbar fails to stabilize the ladder, we shall add one more bar.
ii. To the 9 bar ladder of (i) add a diagonal cross bar between nodes 1 and the left end of bar 6. Compute bases for the column and null spaces of the new adjacency matrix.

Exercise 3.6.2

We wish to show that $N(A) = N(A^T A)$ regardless of A.

i. We first take a concrete example. Report the findings of `null` when applied to A and $A^T A$ for the A matrix associated with the unstable ladder figure.
ii. Show that $N(A) \subseteq N(A^T A)$ i.e. that if $A\mathbf{x} = \mathbf{0}$ then $A^T A\mathbf{x} = \mathbf{0}$.
iii. Show that $N(A^T A) \subseteq N(A)$ i.e., that if $A^T A\mathbf{x} = \mathbf{0}$ then $A\mathbf{x} = \mathbf{0}$ (Hint: if $A^T A\mathbf{x} = \mathbf{0}$ then $\mathbf{x}^T A^T A\mathbf{x} = \mathbf{0}$

Exercise 3.6.3

Suppose that A is m-by-n and that $N(A) = \mathbb{R}^n$. Argue that A must be the zero matrix.

3.7: Supplements - Vector Space

Introduction

You have long taken for granted the fact that the set of real numbers, $\mathbb{R}$, is closed under addition and multiplication, that each number has a unique additive inverse, and that the commutative, associative, and distributive laws were right as rain. The set $\mathbb{C}$, of complex numbers also enjoys each of these properties, as do the sets $\mathbb{R}^n$ and $\mathbb{C}^n$ of columns of n real and complex numbers, respectively.

To be more precise, we write $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ as

$\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$

$\mathbf{y} = (y_1, y_2, \cdots, y_n)^T$

and define their vector sum as the elementwise sum

$$\mathbf{x}+\mathbf{y} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}$$

and similarly, the product of a complex scalar, $\mathbf{z} \in \mathbb{C}$ with $\mathbf{x}$ as:

$$\mathbf{zx} = \begin{pmatrix} zx_1 \\ zx_2 \\ \vdots \\ zx_n \end{pmatrix}$$

Vector Space

These notions lead naturally to the concept of vector space. A set V is said to be a vector space if

1. $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for each $\mathbf{x}$ and $\mathbf{y}$ in V.
2. $\mathbf{x}+\mathbf{y}+\mathbf{z} = \mathbf{y}+\mathbf{x}+\mathbf{z}$ for each $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in V.
3. There is a unique "zero vector" such that $\mathbf{x}+\mathbf{0} = \mathbf{x}$ for each $\mathbf{x}$ in V.
4. For each $\mathbf{x}$ in V there is a unique vector $-\mathbf{x}$ such that $\mathbf{x}+-\mathbf{x} = \mathbf{0}$.
5. $1\mathbf{x} = \mathbf{x}$.
6. $(c_1 c_2)\mathbf{x} = c_1(c_2\mathbf{x})$ for each $\mathbf{x}$ in V and c_1 and c_2 in $\mathbb{C}$.
7. $c(\mathbf{x}+\mathbf{y}) = c\mathbf{x}+c\mathbf{y}$ for each $\mathbf{x}$ and $\mathbf{y}$ in V and c in $\mathbb{C}$.
8. $(c_1+c_2)\mathbf{x} = c_1\mathbf{x}+c_2\mathbf{x}$ for each $\mathbf{x}$ in V and c_1 and c_2 in $\mathbb{C}$.

3.8: Supplements - Subspaces

Subspace

A subspace is a subset of a vector space that is itself a vector space. The simplest example is a line through the origin in the plane. For the line is definitely a subset and if we add any two vectors on the line we remain on the line and if we multiply any vector on the line by a scalar we remain on the line. The same could be said for a line or plane through the origin in 3 space. As we shall be travelling in spaces with many many dimensions it pays to have a general definition.

Definition: Subspace

A subset S of a vector space V is a **subspace** of V when

- if x and y belong to S then so does $x+y$
- if x belongs to S and t is real then tx belong to S

As these are oftentimes unwieldy objects it pays to look for a handful of vectors from which the entire subset may be generated. For example, the set of x for which $x_1+x_2+x_3+x_4=0$ constitutes a subspace of $\mathbb{R}^4$. Can you 'see' this set? Do you 'see' that

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

not only belong to a set but in fact generate all possible elements? More precisely, we say that these vectors **span** the subspace of all possible solutions.

Definition: Span

A finite collection $\{s_1, s_2, \cdots, s_n\}$ of vectors in the subspace S is said to **span** S if each element of S can be written as a linear combination of these vectors. That is, if for each $s \in S$ there exist nn reals $\{x_1, x_2, \cdots, x_n\}$ such that $s = x_1 s_1 + x_2 s_2 + \cdots + x_n s_n$.

When attempting to generate a subspace as the span of a handful of vectors it is natural to ask what is the fewest number possible. The notion of linear independence helps us clarify this issue.

Definition: Linear Independence

A finite collection $\{s_1, s_2, \cdots, s_n\}$ of vectors is said to be **linearly independent** when the only reals, $\{x_1, x_2, \cdots, x_n\}$ for which $x_1 + x_2 + \cdots + x_n = 0$ are $x_1 = x_2 = \cdots = x_n = 0$ In other words, when the null space of the matrix whose columns are $\{s_1, s_2, \cdots, s_n\}$ contains only the zero vector.

Combining these definitions, we arrive at the precise notion of a 'generating set.'

Definition: Basis

Any linearly independent spanning set of a subspace S is called a **basis** of S

Though a subspace may have many bases they all have one thing in common:

Definition: Dimension

The **dimension** of a subspace is the number of elements in its basis.

3.9: Supplements - Row Reduced Form

Row Reduction

A central goal of science and engineering is to reduce the complexity of a model without sacrificing its integrity. Applied to matrices, this goal suggests that we attempt to eliminate nonzero elements and so 'uncouple' the rows. In order to retain its integrity the elimination must obey two simple rules.

Elementary Row Operations

- You may swap any two rows.
- You may add to a row a constant multiple of another row.

With these two elementary operations one can systematically eliminate all nonzeros below the diagonal. For example, given

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

it seems wise to swap the first and fourth rows and so arrive at

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

adding the first row to the third now produces

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

subtracting twice the second row from the third yields

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a matrix with zeros below its diagonal. This procedure is not restricted to square matrices. For example, given

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 4 & 2 \\ 3 & 5 & 5 & 3 \end{pmatrix}$$

we start at the bottom left then move up and right. Namely, we subtract 3 times the first row from the third and arrive at

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 4 & 2 \\ 0 & 2 & 2 & 0 \end{pmatrix}$$

and then subtract twice the first row from the second,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix}$$

and finally subtract the second row from the third,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It helps to label the before and after matrices.

Definition: The Row Reduced Form

Given the matrix A we apply elementary row operations until each nonzero below the diagonal is eliminated. We refer to the resulting matrix as A_{red}.

Uniqueness and Pivots

As there is a certain amount of flexibility in how one carries out the reduction it must be admitted that the reduced form is not unique. That is, two people may begin with the same matrix yet arrive at different reduced forms. The differences however are minor, for both will have the same number of nonzero rows and the nonzeros along the diagonal will follow the same pattern. We capture this pattern with the following suite of definitions,

Definition: Pivot Row

Each nonzero row of A_{red} is called a **pivot row.**

Definition: Pivot

The first nonzero term in each row of A_{red} is called a **pivot**.

Definition: Pivot Column

Each column of A_{red} that contains a pivot is called a **pivot column.**

Definition: Rank

The number of pivots in a matrix is called the **rank** of that matrix.

Regarding our example matrices, the first has rank 4 and the second has rank 2.

Row Reduction in MATLAB

MATLAB's `rref` command goes full-tilt and attempts to eliminate ALL off diagonal terms and to leave nothing but ones on the diagonal. I recommend you try it on our two examples. You can watch its individual decisions by using `rrefmovie` instead.

CHAPTER OVERVIEW

4: Least Squares

4.1: Least Squares

Introduction

We learned in the previous chapter that $Ax = b$ need not possess a solution when the number of rows of A exceeds its rank, i.e., $r < m$. As this situation arises quite often in practice, typically in the guise of 'more equations than unknowns,' we establish a rationale for the absurdity $Ax = b$.

The Normal Equations

The goal is to choose x such that Ax is as close as possible to b. Measuring closeness in terms of the sum of the squares of the components we arrive at the 'least squares' problem of minimizing

res

$$(||Ax - b||)^2 = (Ax - b)^T (Ax - b)$$

over all $x \in \mathbb{R}$. The path to the solution is illuminated by the Fundamental Theorem. More precisely, we write

$\forall b_R, b_N, b_R \in \mathbb{R}(A) \wedge b_N \in \mathbb{N}(A^T) : (b = b_R + b_N)$. On noting that (i) $\forall b_R, x \in \mathbb{R}^n : ((Ax - bR) \in \mathbb{R}(A))$ and (ii) $\mathbb{R}(A) \perp \mathbb{N}(A^T)$ we arrive at the Pythagorean Theorem.

Definition: Pythagoream Theorem

$$norm^2(Ax - b) = (||Ax - b_R + b_N||)^2$$
$$= (||Ax - b_R||)^2 + (||b_N||)^2$$

It is now clear from the Pythagorean Theorem that the best x is the one that satisfies

$$Ax = b_R$$

As $b_R \in \mathbb{R}(A)$ this equation indeed possesses a solution. We have yet however to specify how one computes b_R given b. Although an explicit expression for b_R **orthogonal projection** of b onto $\mathbb{R}(A)$, in terms of A and b is within our grasp we shall, strictly speaking, not require it. To see this, let us note that if x satisfies the above equation then **orthogonal projection** of b onto $\mathbb{R}(A)$, in terms of A and b is within our grasp we shall, strictly speaking, not require it. To see this, let us note that if x satisfies the above equation then

$$Ax - b = Ax - b_R + b_N$$
$$= -b_N$$

As b_N is no more easily computed than b_R you may claim that we are just going in circles. The 'practical' information in the above equation however is that $(Ax - b) \in A^T$, i.e., $A^T(Ax - b) = 0$, i.e.,

$$A^T Ax = A^T b$$

As $A^T b \in \mathbb{R}(A^T)$ regardless of b this system, often referred to as the **normal equations**, indeed has a solution. This solution is unique so long as the columns of $A^T A$ are linearly independent, i.e., so long as $\mathbb{N}(A^T A) = 0$. Recalling Chapter 2, Exercise 2, we note that this is equivalent to $\mathbb{N}(A) = \{0\}$

The set of $x \in b_R$ for which the misfit $(||Ax - b||)^2$ is smallest is composed of those x for which $A^T Ax = A^T b$ There is always at least one such x. There is exactly one such x if $\mathbb{N}(A) = \{0\}$.

As a concrete example, suppose with reference to Figure 1 that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

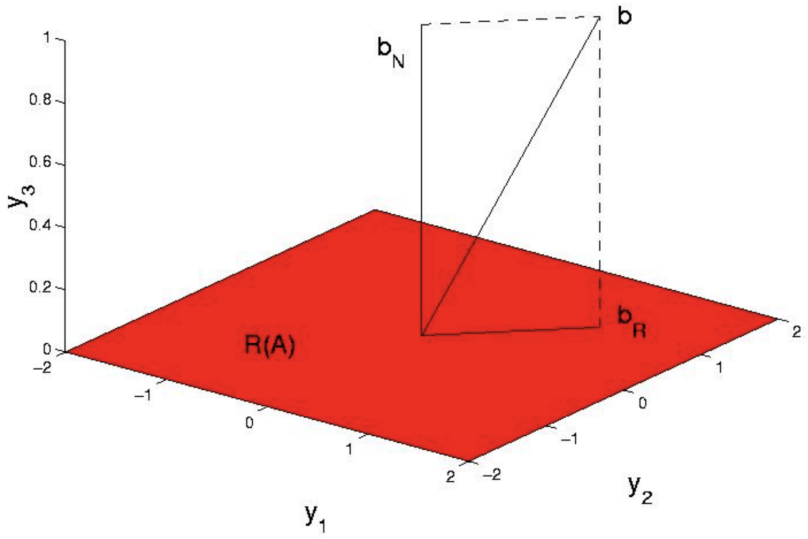

Figure 4.1.1: The decomposition of b

As $b \neq \mathbb{R}(A)$ there is no x such that $Ax = b$. Indeed, $(||Ax - b||)^2 = (x_1 + x_2 + -1)^2 + (x_2 - 1)^2 + 1 \geq 1$, with the minimum uniquely attained at $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, in agreement with the unique solution of the above equation, for $A^T A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $A^T b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We now recognize, a posteriori, that $b_R = Ax = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is the orthogonal projection of b onto the column space of A.

Applying Least Squares to the Biaxial Test Problem

We shall formulate the identification of the 20 fiber stiffnesses in this previous figure, as a least squares problem. We envision loading, the 9 nodes and measuring the associated 18 displacements, x. From knowledge of x and f we wish to infer the components of $K = diag(k)$ where k is the vector of unknown fiber stiffnesses. The first step is to recognize that

$$A^T K A x = f$$

may be written as

$$\forall B, B = A^T diag(Ax) : (Bk = f)$$

Though conceptually simple this is not of great use in practice, for B is 18-by-20 and hence the above equation possesses many solutions. The way out is to compute k as the result of more than one experiment. We shall see that, for our small sample, 2 experiments will suffice. To be precise, we suppose that x^1 is the displacement produced by loading f^1 while x^2 is the displacement produced by loading f^2. We then piggyback the associated pieces in

$$B = \begin{pmatrix} A^T \text{diag}(Ax^1) \\ A^T \text{diag}(Ax^2) \end{pmatrix} \tag{4.1.1}$$

and

$$f = \begin{pmatrix} f^1 \\ f^2 \end{pmatrix}. \tag{4.1.2}$$

This B is 36-by-20 and so the system $Bk = f$ is overdetermined and hence ripe for least squares.

We proceed then to assemble B and f. We suppose f^1 and f^2 to correspond to horizontal and vertical stretching

$$f^1 = (-1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0)^T$$

$$f^2 = (0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0 \quad -1 \quad 0 \quad -1)^T$$

respectively. For the purpose of our example we suppose that each $k_j = 1$ except $k_8 = 5$. We assemble $A^T K A$ as in Chapter 2 and solve

$$A^T K A x^j = f^j$$

with the help of the pseudoinverse. In order to impart some `reality' to this problem we taint each x^j with 10 percent noise prior to constructing B

$$B^T B k = B^T f$$

we note that Matlab solves this system when presented with `k=B\f` when BB is rectangular. We have plotted the results of this procedure in the link. The stiff fiber is readily identified.

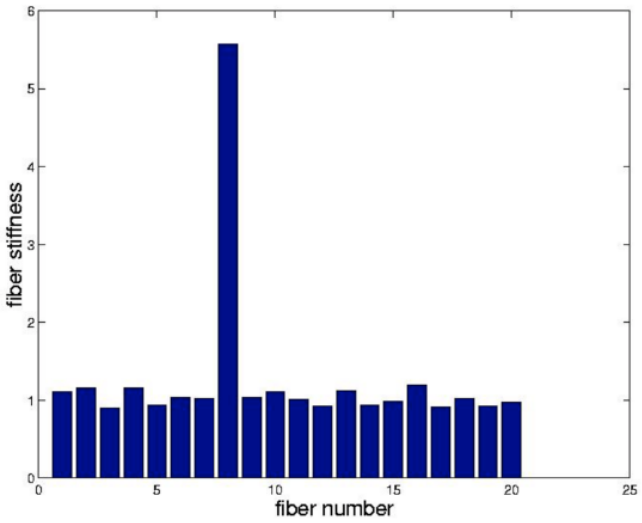

Figure 4.1.1: Results of a successful biaxial test.

Projections

From an algebraic point of view Equation is an elegant reformulation of the least squares problem. Though easy to remember it unfortunately obscures the geometric content, suggested by the word 'projection,' of Equation. As projections arise frequently in many applications we pause here to develop them more carefully. With respect to the normal equations we note that if $\mathbb{N}(A) = \{0\}$ then

$$x = (A^T A)^{-1} A^T b$$

and so the orthogonal projection of bb onto $\mathbb{R}(A)$ is:

$$b_R = Ax$$
$$= A(A^T A)^{-1} A^T b$$

Defining

$$P = A(A^T A)^{-1} A^T$$

takes the form $b_R = Pb$. Commensurate with our notion of what a 'projection' should be we expect that P map vectors not in $\mathbb{R}(A)$ onto $\mathbb{R}(A)$ while leaving vectors already in $\mathbb{R}(A)$ unscathed. More succinctly, we expect that $Pb_R = b_R$ i.e., $Pb_R = Pb_R$. As the latter should hold for all $b \in R^m$ we expect that

$$P^2 = P$$

We find that indeed

$$P^2 = A(A^T A)^{-1} A^T A(A^T A)^{-1} A^T$$
$$= A(A^T A)^{-1} A^T$$

$$= P$$

We also note that the P is symmetric. We dignify these properties through

Definition: Orthogonal Projection

A matrix P that satisfies $P^2 = P$ is called a **projection**. A symmetric projection is called an **orthogonal projection**.

We have taken some pains to motivate the use of the word 'projection.' You may be wondering however what symmetry has to do with orthogonality. We explain this in terms of the tautology

$$b = Pb - Ib$$

Now, if P is a projection then so too is $I - P$. Moreover, if P is symmetric then the dot product of b.

\[

$$\begin{aligned}(Pb)^T(I-P)b &= b^T P^T (I-P)b \\ &= b^T(P-P^2)b \\ &= b^T 0 b \\ &= 0\end{aligned}$$

i.e., Pb is orthogonal to $(I-P)b$. As examples of a nonorthogonal projections we offer

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & 0 & 0 \\ \frac{-1}{4} & \frac{-1}{2} & 1 \end{pmatrix} \tag{4.1.3}$$

and $I - P$. Finally, let us note that the central formula $P = A(A^T A)^{-1} A^T$, is even a bit more general than advertised. It has been billed as the orthogonal projection onto the column space of A. The need often arises however for the orthogonal projection onto some arbitrary subspace M. The key to using the old PP is simply to realize that **every** subspace is the column space of some matrix. More precisely, if

$$\{x_1, \cdots, x_m\}$$

is a basis for MM then clearly if these x_j are placed into the columns of a matrix called A then $\mathbb{R}(A) = M$. For example, if M is the line through $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$ then

$$P = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$P = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is orthogonal projection onto M.

Exercises

Exercise 4.1.1

Gilbert Strang was stretched on a rack to lengths $l = 6, 7, 8$ feet under applied forces of $f = 1, 2, 4$ tons. Assuming Hooke's law $l - L = cf$, find his compliance, c, and original height, L, by least squares.

Exercise 4.1.2

With regard to the example of § 3 note that, due to the the random generation of the noise that taints the displacements, one gets a different 'answer' every time the code is invoked.

1. Write a loop that invokes the code a statistically significant number of times and submit bar plots of the average fiber stiffness and its standard deviation for each fiber, along with the associated M--file.
2. Experiment with various noise levels with the goal of determining the level above which it becomes difficult to discern the stiff fiber. Carefully explain your findings.

Exercise 4.1.3

Find the matrix that projects $\mathbb{R}^3$ onto the line spanned by $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$.

Exercise 4.1.4

Find the matrix that projects $\mathbb{R}^3$ onto the line spanned by $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$ and $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}^T$.

Exercise 4.1.5

If P is the projection of $\mathbb{R}^m$ onto a k--dimensional subspace M, what is the rank of P and what is $\mathbb{R}(P)$?

CHAPTER OVERVIEW

5: Matrix Methods for Dynamical Systems

5.1: Nerve Fibers and the Dynamic Strang Quartet

Introduction

Up to this point we have largely been concerned with

1. Deriving linear systems of algebraic equations (from considerations of static equilibrium) and
2. The solution of such systems via Gaussian elimination.

In this module we hope to begin to persuade the reader that our tools extend in a natural fashion to the class of dynamic processes. More precisely, we shall argue that

1. Matrix Algebra plays a central role in the derivation of mathematical models of dynamical systems and that,
2. With the aid of the Laplace transform in an analytical setting or the Backward Euler method in the numerical setting, Gaussian elimination indeed produces the solution.

Nerve Fibers and the Dynamic Strang Quartet

Gathering Information

A nerve fiber's natural electrical stimulus is not direct current but rather a short burst of current, the so-called **nervous impulse**. In such a dynamic environment the cell's membrane behaves not only like a leaky conductor but also like a charge separator, or **capacitor**.

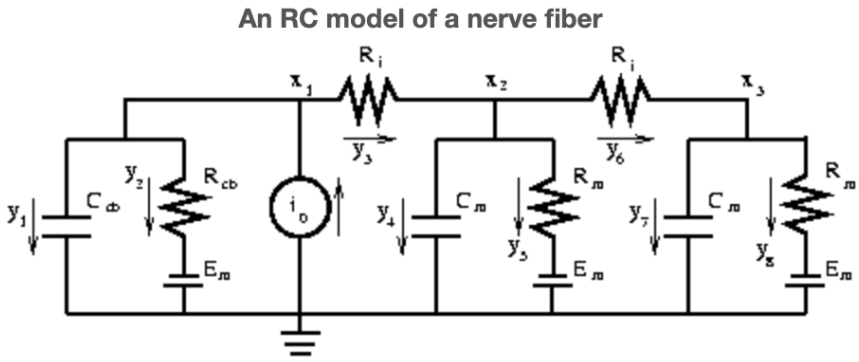

Figure 1.

The typical value of a cell's membrane capacitance is

$$c = 1 \frac{\mu F}{cm^2}$$

where μF denotes micro-Farad. Recalling our variable conventions, the capacitance of a single compartment is

$$C_m = 2\pi a \frac{l}{N} c$$

and runs parallel to each R_m, see Figure 1. This figure also differs from the simpler circuit from the introductory electrical modeling module in that it possesses two edges to the left of the stimuli. These edges serve to mimic that portion of the stimulus current that is shunted by the cell body. If A_{cb} denotes the surface area of the cell body, then it has

Definition: Capacitance of Cell Body

$C_{cb} = A_{cb} c$

Definition: Resistance of Cell Body

$R_{cb} = A_{cb}\rho_m$

Updating the Strang Quartet

We ask now how the static Strang Quartet of the introductory electrical module should be augmented.

Updating (S1')

Regarding (S1') we proceed as before. The voltage drops are

$$e_1 = x_1$$
$$e_2 = x_1 - E_m$$
$$e_3 = x_1 - x_2$$
$$e_4 = x_2$$
$$e_5 = x_2 - E_m$$
$$e_6 = x_2 - x_3$$
$$e_7 = x_3$$
$$e_8 = x_3 - E_m$$

and so

$$\mathbf{e} = \mathbf{b} - A\mathbf{x} \quad \text{where} \quad \mathbf{b} = (-E_m)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Updating (S2)

To update (S2) we must now augment Ohm's law with

Definition: Voltage-current Law Obeyed by a Capacitor

The current through a capacitor is proportional to the time rate of change of the potential across it.

This yields, (denoting derivative by '),

$$y_1 = C_{cb}e_1{}'$$
$$y_2 = \frac{e_2}{R_{cb}}$$
$$y_3 = \frac{e_3}{R_i}$$
$$y_4 = C_m e_4{}'$$
$$y_5 = \frac{e_5}{R_m}$$
$$y_6 = \frac{e_6}{R_i}$$

$$y_7 = C_m {e_7}'$$

$$y_8 = \frac{e_8}{R_m}$$

or, in matrix terms,

$$\mathbf{y} = G\mathbf{e} + C\mathbf{e}'$$

where

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_{cb}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R_m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_m} \end{pmatrix}$$

and

$$C = \begin{pmatrix} C_{cb} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

are the conductance and capacitance matrices.

Updating (S3)

As Kirchhoff's Current law is insensitive to the type of device occupying an edge, step (S3) proceeds exactly as before.

$$i_0 - y_1 - y_2 - y_3 = 0$$

$$y_3 - y_4 - y_5 - y_6 = 0$$

$$y_6 - y_7 - y_8 = 0$$

or, in matrix terms,

$$A^T\mathbf{y} = -\mathbf{f} \quad where \quad \mathbf{f} = \begin{pmatrix} i_0 \\ 0 \\ 0 \end{pmatrix}^T$$

Step (S4): Assembling

Step (S4) remains one of assembling,

$$(A^T\mathbf{y} = -\mathbf{f}) \Rightarrow (A^T(G\mathbf{e} + C\mathbf{e}') = -\mathbf{f}) \Rightarrow (A^T(G(\mathbf{b} - A\mathbf{x}) + C(\mathbf{b}' - A\mathbf{x}')) = -\mathbf{f})$$

becomes

$$A^TCA\mathbf{x}' + A^TGA\mathbf{x} = A^TG\mathbf{b} + \mathbf{f} + A^TC\mathbf{b}'$$

This is the general form of the potential equations for an RC circuit. It presumes of the user knowledge of the initial value of each of the potentials,

$$\mathbf{x}(0) = X$$

Regarding the circuit of Figure 1, and letting $G = \frac{1}{R}$, we find

$$A^T C A = \begin{pmatrix} C_{cb} & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix} \quad A^T C A = \begin{pmatrix} G_{cb}+G_i & -G_i & 0 \\ -G_i & 2G_i+G_m & -G_i \\ 0 & -G_i & G_i+G_m \end{pmatrix}$$

$$A^T G\mathbf{b} = E_m \begin{pmatrix} G_{cb} \\ G_m \\ G_m \end{pmatrix} \quad A^T C\mathbf{b}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and an initial (rest) potential of

$$x_0 = E_m \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^T$$

Modes of Attack

We shall now outline two modes of attack on such problems. The Laplace Transform is an analytical tool that produces exact, closed-form solutions for small tractable systems and therefore offers insight into how larger systems 'should' behave. The Backward-Euler method is a technique for solving a discretized (and therefore approximate) version of Equation. It is highly flexible, easy to code, and works on problems of great size. Both the Backward-Euler and Laplace Transform methods require, at their core, the algebraic solution of a linear system of equations. In deriving these methods we shall find it more convenient to proceed from the generic system

$$\mathbf{x}' = B\mathbf{x} + \mathbf{g}$$

With respect to our fiber problem

$$B = (-(A^T C A)^{-1}) A^T G A$$

$$= \begin{pmatrix} \frac{-(G_{cb}+G_i)}{C_{cb}} & \frac{G_i}{C_{cb}} & 0 \\ \frac{G_i}{C_m} & \frac{-(2G_i+G_m)}{C_m} & \frac{G_i}{C_m} \\ 0 & \frac{G_i}{C_m} & \frac{-(G_i+G_m)}{C_m} \end{pmatrix}$$

and

$$\mathbf{g} = (A^T C A)^{-1}(A^T G\mathbf{b} + \mathbf{f}) = \begin{pmatrix} \frac{G_{cb}E_m + i_0}{C_{cb}} \\ \frac{E_m G_m}{C_m} \\ \frac{E_m G_m}{C_m} \end{pmatrix}$$

5.2: The Laplace Transform

The Laplace Transform is typically credited with taking dynamical problems into static problems. Recall that the Laplace Transform of the function h is

$$\mathscr{L}(h(s)) \equiv \int_0^\infty e^{-(st)} h(t)dt$$

MATLAB is very adept at such things. For example:

The Laplace Transform in MATLAB

```
>> syms t

>> laplace(exp(t))

ans = 1/(s-1)

>> laplace(t*(exp(-t))

ans = 1/(s+1)^2
```

The Laplace Transform of a matrix of functions is simply the matrix of Laplace transforms of the individual elements.

Definition: Laplace Transform of a matrix of fucntions

$$\mathscr{L}(\begin{pmatrix} e^t \\ te^{-t} \end{pmatrix}) = \begin{pmatrix} \frac{1}{s-1} \\ \frac{1}{(s+1)^2} \end{pmatrix}$$

Now, in preparing to apply the Laplace transform to our equation from the dynamic strang quartet module:

$$\mathbf{x}' = B\mathbf{x} + \mathbf{g} \tag{5.2.1}$$

we write it as

$$\mathscr{L}(\frac{dx}{dt}) = \mathscr{L}(B\mathbf{x} + \mathbf{g}) \tag{5.2.2}$$

and so must determine how $\mathscr{L}$ acts on derivatives and sums. With respect to the latter it follows directly from the definition that

$$\begin{aligned} \mathscr{L}(B\mathbf{x} + \mathbf{g}) &= \mathscr{L}(B\mathbf{x}) + \mathscr{L}(\mathbf{g}) \\ &= B\mathscr{L}(\mathbf{x}) + \mathscr{L}(\mathbf{g}) \end{aligned}$$

Regarding its effect on the derivative we find, on integrating by parts, that

$$\mathscr{L}\left(\frac{d\mathbf{x}}{dt}\right) = \int_0^\infty e^{-(st)} \frac{d\mathbf{x}(t)}{dt} dt \tag{5.2.3}$$

$$= \mathbf{x}(t)e^{-(st)}\Big|_0^\infty + s\int_0^\infty e^{-(st)}\mathbf{x}(t)dt \tag{5.2.4}$$

Supposing that x and s are such that $x(t)e^{-(st)} \to 0$ as $t \to \infty$ we arrive at

$$\mathscr{L}(\frac{d\mathbf{x}}{dt}) = s\mathscr{L}(\mathbf{x}) - x(0)$$

Now, upon substituting Equation 2 and Equation 3 into Equation 1 we find

$$s\mathscr{L}(\mathbf{x}) - \mathbf{x}(0) = B\mathscr{L}(\mathbf{x}) + \mathscr{L}(\mathbf{g})$$

which is easily recognized to be a linear system for $\mathscr{L}(\mathbf{x})$

$$(\mathbf{s}I - B)\mathscr{L}(\mathbf{x}) = \mathscr{L}(\mathbf{g}) + x(0)$$

The only thing that distinguishes this system from those encountered since our first brush with these systems is the presence of the complex variable s. This complicates the mechanical steps of Gaussian Elimination or the Gauss-Jordan Method but the methods indeed apply without change. Taking up the latter method, we write

$$\mathscr{L}(\mathbf{x}) = (sI - B)^{-1}(\mathscr{L}(\mathbf{g}) + x(0))$$

The matrix $(sI - B)^{-1}$ is typically called the **transfer function** or **resolvent**, associated with B, at s. We turn to MATLAB for its **symbolic** calculation. (for more information, see the tutorial on MATLAB's symbolic toolbox). For example,

```
>> B = [2 -1; -1 2]

>> R = inv(s*eye(2)-B)

R =

[ (s-2)/(s*s-4*s+3), -1/(s*s-4*s+3)]

[ -1/(s*s-4*s+3), (s-2)/(s*s-4*s+3)]
```

We note that $(sI - B)^{-1}$ well defined except at the roots of the quadratic, $s^2 - 4s + 3$ **determinant** of $(sI - B)$ and is often referred to as the **characteristic polynomial** of B. Its roots are called the **eigenvalues** of B.

Example 5.2.1

Let us take the B matrix of the dynamic Strang quartet module with the parameter choices specified in fib3.m, namely

$$B = \begin{pmatrix} -0.135 & 0.125 & 0 \\ 0.5 & -1.01 & 0.5 \\ 0 & 0.5 & -0.51 \end{pmatrix}$$

The associated $(sI - B)^{-1}$ is a bit bulky (please run fig3.m) so we display here only the denominator of each term, i.e.,

$$s^3 + 1.655s^2 + 0.4078s + 0.0039$$

Assuming a current stimulus of the form $i_0(t) = \frac{t^3 e^{-\frac{t}{6}}}{10000}$ and $E_m = 0$ brings

$$\mathscr{L}(\mathbf{g})(s) = \begin{pmatrix} \frac{0.191}{(s+\frac{1}{6})^4} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and so Equation persists in

$$\begin{aligned} \mathscr{L}(\mathbf{x}) &= (sI - B)^{-1}\mathscr{L}(\mathbf{g}) \\ &= \frac{0.191}{(s+\frac{1}{6})^4(s^3 + 1.655s^2 + 0.4078s + 0.0039)} \begin{pmatrix} s^2 + 1.5s + 0.27 \\ 0.5s + 0.26 \\ 0.2497 \end{pmatrix} \end{aligned}$$

Now comes the rub. A simple linear solve (or inversion) has left us with the Laplace transform of $\mathbf{x}$. The accursed **No Free Lunch Theorem**

We shall have to do some work in order to recover $\mathbf{x}$ from $\mathscr{L}(\mathbf{x})$ confronts us. We shall face it down in the Inverse Laplace module.

5.3: The Inverse Laplace Transform

To Come

In The transfer Function we shall establish that the inverse Laplace transform of a function h is

$$\mathscr{L}^{-1}(h)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(c+yi)t}h((c+yi)t)dy$$

where $i \equiv \sqrt{2-1}$ and the real number c is chosen so that all of the **singularities** of h lie to the left of the line of integration.

Proceeding with the Inverse Laplace Transform

With the inverse Laplace transform one may express the solution of $\mathbf{x}' = B\mathbf{x} + \mathbf{g}$, as

$$x(t) = \mathscr{L}^{-1}((sI - B)^{-1})(\mathscr{L}(\mathbf{g} + x(0))$$

As an example, let us take the first component of $\mathscr{L}(x)$, namely

$$\mathscr{L}_{x_1}(s) = \frac{0.19(s^2 + 1.5s + 0.27)}{(s+\frac{1}{6})^4(s^3 + 1.655s^2 + 0.4078s + 0.0039)}$$

We define:

Definition: Pole

Also called singularities, these are the points ss at which $\mathscr{L}_{x_1}(s)$ blows up.

These are clearly the roots of its denominator, namely

$$-1/100 \quad -329/400 \pm \frac{\sqrt{273}}{16} \quad and \quad -1/6$$

All four being negative, it suffices to take $c = 0$ and so the integration in Equation proceeds up the imaginary axis. We don't suppose the reader to have already encountered integration in the complex plane but hope that this example might provide the motivation necessary for a brief overview of such. Before that however we note that MATLAB has digested the calculus we wish to develop. Referring again to fib3.m for details we note that the `ilaplace` command produces

$$x_1(t) = 211.35e^{\frac{-t}{100}} - (0.0554t^3 + 4.5464t^2 + 1.085t + 474.19)e^{\frac{-t}{6}} + e^{\frac{-(329t)}{400}}(262.842\cosh(\frac{\sqrt{273}t}{16})) + 262.836\sinh(\frac{\sqrt{273}t}{16}))$$

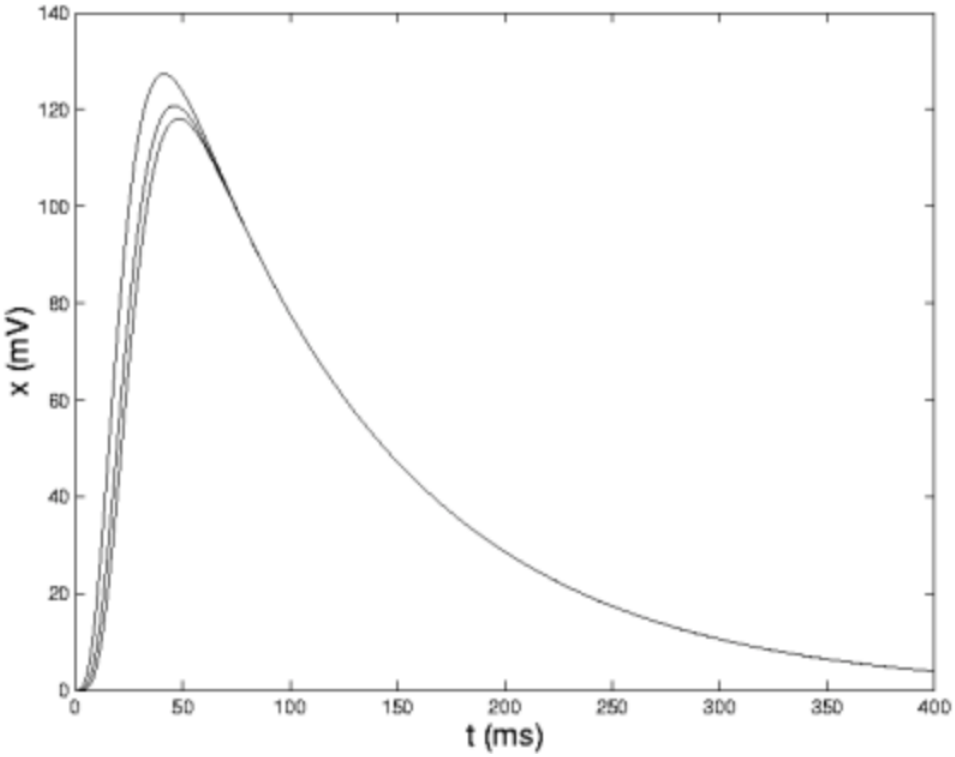

Figure 1. The 3 potentials associated with the RC circuit model figure.

The other potentials, see the figure above, possess similar expressions. Please note that each of the poles of $\mathscr{L}(x_1)$ appear as exponents in x_1 and that the coefficients of the exponentials are polynomials whose degrees is determined by the **order** of the respective pole.

5.4: The Backward-Euler Method

Where in the Inverse Laplace Transform section we tackled the derivative in

$$\mathbf{x} = B\mathbf{x} + \mathbf{g}$$

via an integral transform we pursue in this section a much simpler strategy, namely, replace the derivative with a finite difference quotient. That is, one chooses a small dt and 'replaces' Equation with

$$\frac{\tilde{x}(t) - \tilde{x}(t-dt)}{dt} = B\tilde{x}(t) + g(t)$$

The utility of Equation is that it gives a means of solving for $\tilde{x}$ at the present time, t, from the knowledge of $\tilde{x}$ in the immediate past, $t - dt$.

For example, as $\tilde{x}(0) = x(0)$ is supposed known we write Equation as $\tilde{x}(0) = x(0)$ is supposed known we write Equation as

$$(\frac{I}{dt} - B)\tilde{x}(dt) = x(0)dt + g(dt)$$

Solving this for $\tilde{x}(dt)$ we return to Equation and find

$$(\frac{I}{dt} - B)\tilde{x}(2dt) = x(dt)dt + g(2dt)$$

and solve for $\tilde{x}(2dt)$. The general step from past to present,

$$\tilde{x}(jdt) = (\frac{I}{dt} - B)^{-1}(\frac{\tilde{x}((j-1)dt)}{dt} + g(jdt))$$

is repeated until some desired final time, Tdt is reached. This equation has been implemented in fib3.m with $dt = 1$ and B and g as in the dynamic Strang module. The resulting $\tilde{x}$ (run fib3.m yourself!) is indistinguishable from the plot we obtained in the Inverse Laplace module.

Comparing the two representations, we see that they both produce the solution to the general linear system of ordinary equations by simply inverting a shifted copy of B. The former representation is hard but exact while the latter is easy but approximate. Of course we should expect the approximate solution, $\tilde{x}$, to approach the exact solution, x, as the time step dt, approaches zero. To see this let us return to Equation and assume, for now, that $g \equiv 0$. In this case, one can reverse the above steps and arrive at the representation

$$\tilde{x}(jdt) = ((I - dtB)^{-1})^j x(0)$$

Now, for a fixed time t we suppose that $dt = \frac{t}{j}$ and ask whether

$$x(t) = \lim_{j\to\infty}((I - \frac{t}{j}B)^{-1})^j x(0)$$

This limit, at least when B is one-by-one, yields the exponential

$$x(t) = e^{Bt}x(0)$$

clearly the correct solution to the equation. A careful explication of the **matrix exponential** and its relationship to the transfer function will have to wait until we have mastered the inverse laplace transform.

5.5: Exercises- Matrix Methods for Dynamical Systems

Exercise 5.5.1

Compute, **without** the aid of a machine, the Laplace transforms of e^t and te^{-t}. Show **ALL** of your work.

Exercise 5.5.2

Extract from `fib3.m` analytical expressions for x_2 and x_3

Exercise 5.5.3

Use `eig` to compute the eigenvalues of $B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Use `det` to compute the characteristic polynomial of B `roots` to compute the roots of this characteristic polynomial. Compare these to the results of `eig` . How does Matlab compute the roots of a polynomial? (type `help roots` for the answer).

Exercise 5.5.4

Adapt the Backward Euler portion of `fib3.m` so that one may specify an arbitrary number of compartments, as in `fib1.m` . Submit your well documented M-file along with a plot of x_1 and x_{10} **versus** time (on the same well labeled graph) for a nine compartment fiber of length $l = 1cm$.

Exercise 5.5.5

Derive $\frac{\tilde{x}(t)-\tilde{x}(t-dt)}{dt} = B\tilde{x}(t)+g(t)$ from $\mathbf{x}' = B\mathbf{x}+\mathbf{g}$, by working backwards toward $x(0)$. Along the way you should explain why

$$\frac{(\frac{I}{d(t)}-B)^{-1}}{d(t)} = (I-d(t)B)^{-1}$$

5.6: Supplemental - Matrix Analysis of the Branched Dendrite Nerve Fiber

Introduction

In the prior modules on static and dynamic electrical systems, we analyzed basic, hypothetical one-branch nerve fibers using a modeling methodology we dubbed the Strang Quartet. You may be asking yourself whether this method is stout enough to handle the real fiber of our minds. Indeed, can we use our tools in a real-world setting (Figure 5.6.1)?

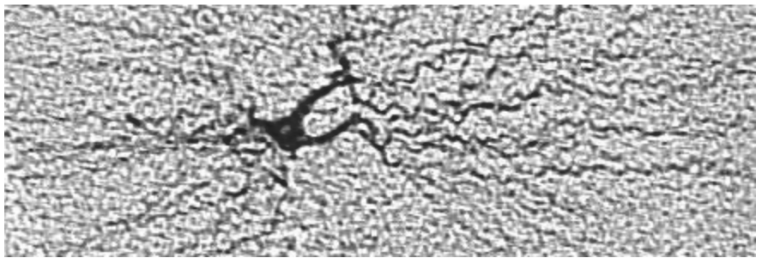

Figure 5.6.1: An Actual Nerve Fiber. A pyramidal neuron from the CA3 region of a rat's hippocampus, scanned at (FIX ME) X magnification.

To answer your question, the above is a rendering of a neuron from a rat's hippocampus. The tools we have refined will enable us to model the electrical properties of a dendrite leaving the neuron's cell body. A three-branch model of such a dendrite, traced out with painstaking accuracy, appears in Figure 5.6.2.

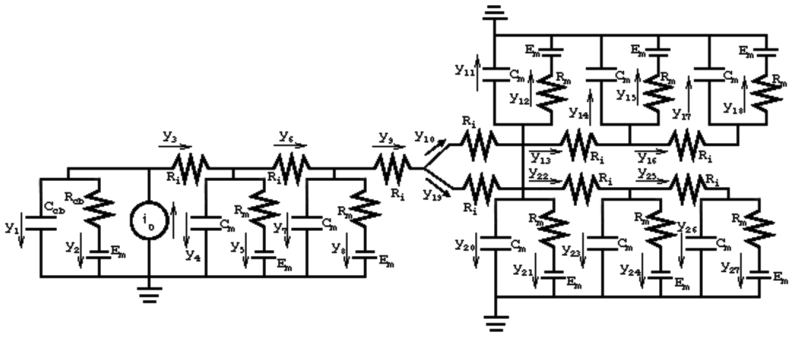

Figure 5.6.2: 3-branch Dendrite Model. Multi-compartment electrical model of a rendered dendrite fiber.

Our multi-compartment model reveals a 3 branch, 10 node, 27 edge structure to the fiber. Note that we have included the Nernst potentials, the nervous impulse as a current source, and the additional leftmost edges depicting stimulus current shunted by the cell body.

We will continue using our previous notation, namely: R_i and R_m denoting cell body. and membrane resistances, respectively; $\mathbf{x}$ representing the vector of potentials $x_1 \cdots x_{10}$, and $\mathbf{x}$ denoting the vector of currents $y_1 \cdots y_{27}$. Using the typical value for a cell's membrane

$$c = 1(\mu F/cm^2)$$

we derive (see variable conventions):

Definition: Capacitance of a Single Compartment

$$C_m = 2\pi a \frac{l}{N} c$$

This capacitance is modeled in parallel with the cell's membrane resistance. Additionally, letting A_{cb} denote the cell body's surface area, we recall that its capacitance and resistance are

Definition: Capacitance of a Cell Body

$$C_{cb} = A_{cb} c$$

Definition: Resistance of a Cell Body

$$C_{cb} = A_{cb}\rho_m$$

Applying the Strang Quartet

Step (S1')--Voltage Drops

Let's begin filling out the Strang Quartet. For Step (S1'), we first observe the voltage drops in the figure. Since there are a whopping 27 of them, we include only the first six, which are slightly more than we need to cover all variations in the set:

$$e_1 = x_1$$

$$e_2 = x_1 - E_m$$

$$e_3 = x_1 - x_2$$

$$e_4 = x_2$$

$$e_5 = x_2 - E_m$$

$$e_6 = x_2 - x_3$$

$$\dots$$

$$e_{27} = x_{10} - E_m$$

In matrix for, letting $\mathbf{b}$ denote the vector of batteries,

$$\mathbf{x} = \mathbf{b} - A\mathbf{x} \quad where \quad \mathbf{b} = (-Em) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$A = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

Although our adjacency matrix A is appreciably larger than our previous examples, we have captured the same phenomena as before.

Applying (S2): Ohm's Law Augmented with Voltage-Current Law for Capacitors

Now, recalling Ohm's Law and remembering that the current through a capacitor varies proportionately with the time rate of change of the potential across it, we assemble our vector of currents. As before, we list only enough of the 27 currents to fully characterize the set:

$$y_1 = C_{cb}\frac{de_1}{dt}$$

$$y_2 = \frac{e_2}{R_{cb}}$$

$$y_3 = \frac{e_3}{R_i}$$

$$y_4 = C_m\frac{de_4}{dt}$$

$$y_5 = \frac{e_5}{R_m}$$

$$\ldots$$

$$y_{27} = \frac{e_{27}}{R_m}$$

In matrix terms, this compiles to

$$\mathbf{y} = G\mathbf{e} + Cd\mathbf{e}dt$$

where

Conductance matrix

$$
G = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{R_{cb}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{R_m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_m} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_m} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R_m}
\end{pmatrix}
$$

and

Capacitance matrix

$$= \overset{C}{\begin{pmatrix}
C_{cb} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_m & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_m & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}}$$

Step (S3): Applying Kirchoff's Law

Our next step is to write out the equations for Kirchoff's Current Law. We see:

$$i_0 - y_1 - y_2 - y_3 = 0$$

$$y_3 - y_4 - y_5 - y_6 = 0$$

$$y_6 - y_7 - y_8 - y_9 = 0$$

$$y_9 - y_{10} - y_{19} = 0$$

$$y_{10} - y_{11} - y_{12} - y_{13} = 0$$

$$y_{13} - y_{14} - y_{15} - y_{16} = 0$$

$$y_{16} - y_{17} - y_{18} - y_{19} = 0$$

$$y_{19} - y_{20} - y_{21} - y_{22} = 0$$

$$y_{22} - y_{23} - y_{24} - y_{25} = 0$$

$$y_{25} - y_{26} - y_{27} = 0$$

Since the B coefficient matrix we'd form here is equal to A^T, we can say in matrix terms:

$$A^T \mathbf{y} = -\mathbf{f}$$

where the vector $\mathbf{f}$ is composed of $f_1 = i_0$ and $f_2 \cdots 27 = 0$

Step (S4): Stirring the Ingredients Together

Step (S4) directs us to assemble our previous toils together into a final equation, which we will then endeavor to solve. Using the process derived in the dynamic Strang module, we arrive at the equation

$$A^TCA\frac{d\mathbf{x}}{dt}+A^TGA\mathbf{x}=A^TG\mathbf{b}+\mathbf{f}+A^TC\frac{d\mathbf{b}}{dt}$$

which is the general form for RC circuit potential equations. As we have mentioned, this equation presumes knowledge of the initial value of each of the potentials, $\mathbf{x}(0)=X$.

Observing our circuit, and letting $\frac{1}{R_{foo}}=G_{foo}$, we calculate the necessary quantities to fill out Equation's pieces (for these calculations, see dendrite.m):

$$A^TCA=\begin{pmatrix} C_{cb} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_m \end{pmatrix}$$

$$A^TGA$$

$$=\begin{pmatrix} G_i+G_{cb} & -G_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -G_i & 2G_i+G_m & -G_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -G_i & 2G_i+G_m & -G_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -G_i & 3G_i & -G_i & 0 & 0 & -G_i & 0 & 0 \\ 0 & 0 & 0 & -G_i & 2G_i+G_m & -G_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -G_i & 2G_i+G_m & -G_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -G_i & G_i+G_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -G_i & 0 & 0 & 0 & 2G_i+G_m & -G_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -G_i & 2G_i+G_m & -G_i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -G_i & G_i+G_m \end{pmatrix}$$

$$A^TG\mathbf{b}=E_m\begin{pmatrix} G_{cb} \\ G_m \\ G_m \\ 0 \\ G_m \\ G_m \\ G_m \\ G_m \\ G_m \\ G_m \end{pmatrix}$$

$$A^TC\frac{d\mathbf{b}}{dt}=\mathbf{0}$$

and an initial (rest) potential of

$$\mathbf{x}(0)=E_m\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Applying the Backward-Euler Method

Since our system is so large, the Backward-Euler method is the best path to a solution. Looking at the matrix A^TCA we observe that it is singular and therefore non-invertible. This singularity arises from the node connecting the three branches of the fiber and prevents us from using the simple equation $\mathbf{x}' = B\mathbf{x} + \mathbf{g}$, we used in earlier Backward-Euler-ings. However, we will see that a modest generalization to our previous form yields Equation:

$$D\mathbf{x}' = E\mathbf{x} + \mathbf{g}$$

capturing the form of our system and allowing us to solve for $\mathbf{x}(t)$ Equation as follows:

$$D\mathbf{x}' = E\mathbf{x} + \mathbf{g}$$

$$D\frac{\tilde{x}(t) - \tilde{x}(t-dt)}{dt} = E\tilde{x}(t) + \mathbf{g}$$

$$(D - Edt)\tilde{x}(t) = D\tilde{x}(t-dt) + \mathbf{g}dt$$

$$\tilde{x}(t) = (D - Edt)^{-1}(\tilde{x}(t-dt) + \mathbf{g}dt)$$

where in our case

$$D = A^TCA$$

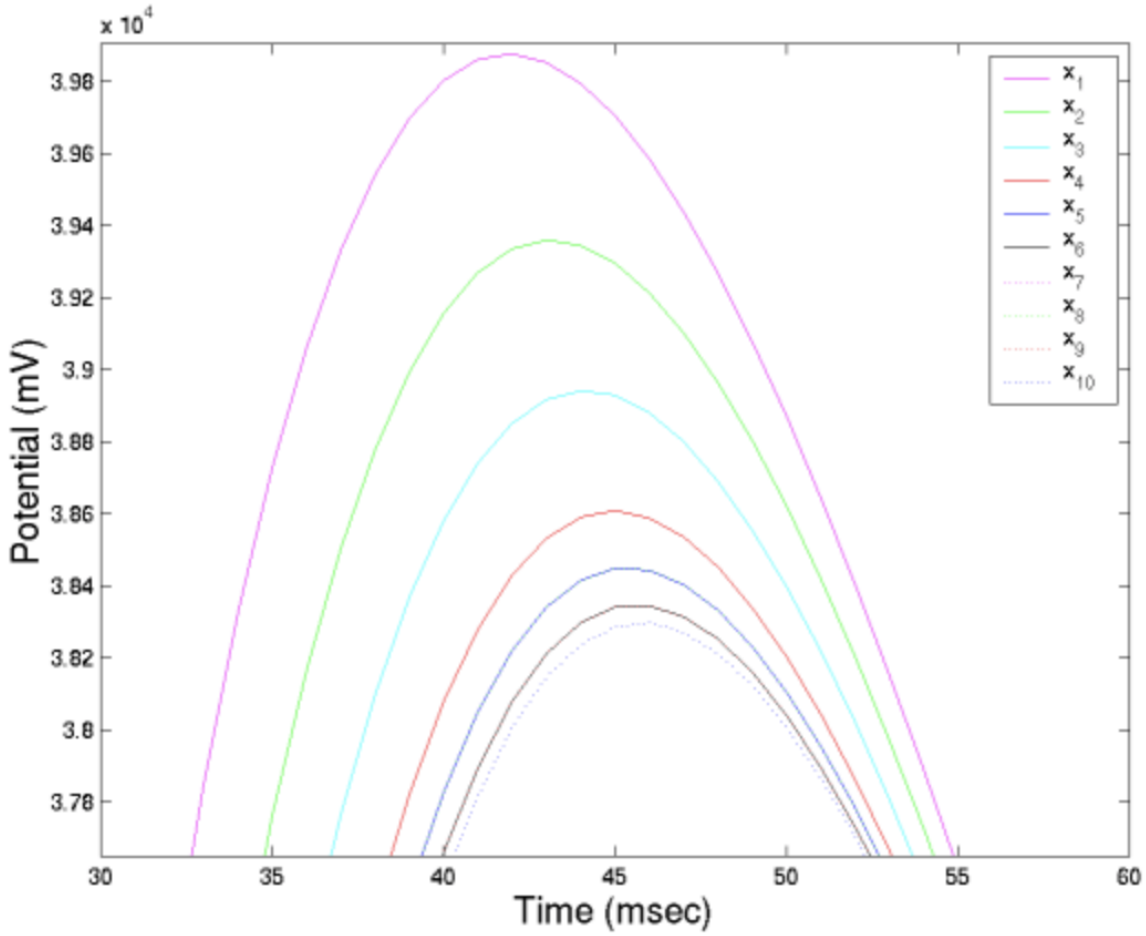
$$E = -(A^TGA)$$

$$\mathbf{g} = A^TG\mathbf{b} + \mathbf{f}$$

.

This method is implemented in dendrite.m with typical cell dimensions and resistivity properties, yielding the following graph of potentials.

Graph of Dendrite Potentials



3b: Zoomed view of potentials showing maxima.

Figure 3.

CHAPTER OVERVIEW

6: Complex Analysis I

6.1: Complex Numbers, Vectors and Matrices

Complex Numbers

A complex number is simply a pair of real numbers. In order to stress however that the two arithmetics differ we separate the two real pieces by the symbol i. More precisely, each complex number, z, may be uniquely expressed by the combination $x+iy$, where x and y are real and i denotes $\sqrt{-1}$. We call x the real part and y the imaginary part of z. We now summarize the main rules of complex arithmetic.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

Definition: Complex Addition

$$z_1 + z_2 \equiv x_1 + x_2 + i(y_1 + y_2)$$

Definition: Complex Multiplication

$$z_1 + z_2 \equiv (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

Definition: Complex Conjugation

$$\overline{z_1} \equiv x_1 - iy_1$$

Definition: Complex Division

$$\frac{z_1}{z_2} \equiv \frac{z_1}{z_2}\frac{\overline{z_2}}{\overline{z_2}} = \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

Definition: Magnitude of a Complex Number

$$|z_1| \equiv= \sqrt{z_1\overline{z_1}} = \sqrt{x_1^2 + y_1^2}$$

Polar Representation

In addition to the Cartesian representation $z = x + iy$ one also has the polar form

$$z = |z|(\cos(\theta) + i\sin(\theta))$$

where $\theta = \arctan(yx)$

This form is especially convenient with regards to multiplication. More precisely,

$$\begin{aligned} z_1z_2 &= |z_1||z_2|(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2))) \\ &= |z_1||z_2|(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

As a result:

$$z^n = (|z|)^n(\cos(n\theta) + i\sin(n\theta))$$

Complex Vectors and Matrices

A complex vector (matrix) is simply a vector (matrix) of complex numbers. Vector and matrix addition proceed, as in the real case, from elementwise addition. The dot or inner product of two complex vectors requires, however, a little modification. This is evident when we try to use the old notion to define the length of a complex vector. To wit, note that if:

$$z = \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$$

then

$$z^T z = (1+i)^2 + (1-i)^2 = 1+2i-1+1-2i-1 = 0$$

Now length **should** measure the distance from a point to the origin and should only be zero for the zero vector. The fix, as you have probably guessed, is to sum the squares of the **magnitudes** of the components of z. This is accomplished by simply conjugating one of the vectors. Namely, we define the length of a complex vector via:

$$(z) = \sqrt{\bar{z}^T z}$$

In the example above this produces

$$\sqrt{(|1+i|)^2 + (|1-i|)^2} = \sqrt{4} = 2$$

As each real number is the conjugate of itself, this new definition subsumes its real counterpart.

The notion of magnitude also gives us a way to define limits and hence will permit us to introduce complex calculus. We say that the sequence of complex numbers, $\left\{ z_n | n = \begin{pmatrix} 1 \\ 2 \\ \dots \end{pmatrix} \right\}$, converges to the complex number z_0 and write

$$z_n \to z_0$$

or

$$z_0 = \lim_{n \to \infty} z_n$$

when, presented with any $\epsilon > 0$ one can produce an integer N for which $|z_n - z_0| < \epsilon$ when $n \geq N$. As an example, we note that $(\frac{i}{2})^n \to 0$.

Example 6.1.1

As an example both of a complex matrix and some of the rules of complex arithmetic, let us examine the following matrix:

$$F = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Let us attempt to find $F\overline{F}$. One option is simply to multiply the two matrices by brute force, but this particular matrix has some remarkable qualities that make the job significantly easier. Specifically, we can note that every element not on the diagonal of the resultant matrix is equal to 0. Furthermore, each element **on** the diagonal is 4. Hence, we quickly arrive at the matrix

$$F\overline{F} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$$= 4i$$

This final observation, that this matrix multiplied by its transpose yields a constant times the identity matrix, is indeed remarkable. This particular matrix is an example of a Fourier matrix, and enjoys a number of interesting properties. The property outlined above can be generalized for any F_n, where F refers to a Fourier matrix with n rows and columns:

$$F_n \overline{F}_n = nI$$

6.2: Complex Functions

Complex Functions

A complex function is merely a rule for assigning certain complex numbers to other complex numbers. The simplest (nonconstant) assignment is the identity function $f(z) \equiv z$ Perhaps the next simplest function assigns to each number its square, i.e., $f(z) \equiv z^2$. As we decomposed the **argument** of f, namely z, **value** of f, z^2 in this case, into its real and imaginary parts. In general, we write

$$f(x+iy) = u(x,y) + iv(x,y)$$

where u and v are both real-valued functions of two real variables. In the case that $f(z) \equiv z^2$ we find

$$u(x,y) = x^2 - y^2$$

and

$$v(x,y) = 2xy$$

With the tools of complex numbers, we may produce complex polynomials

$$f(z) = z^m + c_{m-1}z^{m-1} + \cdots + c_1 z + c_0$$

We say that such an f is order m. We shall often find it convenient to represent polynomials as the product of their factors, namel

$$f(z) = (z-\lambda_1)^{d^1}(z-\lambda_2)^{d^2}\cdots(z-\lambda_h)^{d^h}$$

Each λ_j is a **root** of **degree** d_j. Here h is the number of **distinct** roots of f. We call λ_j a **simple** root when $d_j = 1$ **rational** functions. Suppose

$$q(z) = \frac{f(z)}{g(z)}$$

in rational, that f is of order at most $m-1$ while g is of order m with the simple roots $\{\lambda_1, \cdots, \lambda_m\}$. It should come as no surprise that such q should admit a **Partial Fraction Expansion**

$$q(z) = \sum_{j=1}^{m} \frac{q_j}{z-\lambda_j}$$

One uncovers the q_j by first multiplying each side by $z-\lambda_j$ and then letting z tend to λ_j. For example, if

$$\frac{1}{z^2+1} = \frac{q_1}{z+i} + \frac{q_2}{z-i}$$

then multiplying each side by $z+i$ produces

$$\frac{1}{z-i} = q_1 + \frac{q_2(z+i)}{z-i}$$

Now, in order to isolate q_1 it is clear that we should set $z=-i$. So doing we find that $q_1 = \frac{i}{2}$. In order to find q_2 we multiply Equation by $z-i$ and then set $z=i$. So doing we find $q_2 = -\frac{i}{2}$, and so

$$\frac{1}{z^2+i} = \frac{\frac{i}{2}}{z+i} + \frac{\frac{-i}{2}}{z-i}$$

Returning to the general case, we encode the above in the simple formula

$$q_j = \lim_{zZ\to\lambda_j} (z-\lambda_j)q(z)$$

You should be able to use this to confirm that

$$\frac{z}{z^2+1} = \frac{1/2}{z+i} + \frac{1/2}{z-i}$$

Recall that the transfer function we met in The Laplace Transform module was in fact a matrix of rational functions. Now, the partial fraction expansion of a matrix of rational functions is simply the matrix of partial fraction expansions of each of its elements. This is easier done than said. For example, the transfer function of

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.2.1}$$

is

$$\begin{aligned}(zI - B)^{-1} &= \frac{1}{z^2+1}\begin{pmatrix} z & 1 \\ -1 & z \end{pmatrix} \\ &= \frac{1}{z+i}\begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{-i}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{z-i}\begin{pmatrix} \frac{1}{2} & \frac{-i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}\end{aligned}$$

The first line comes form either Gauss-Jordan by hand or via the symbolic toolbox in Matlab. More importantly, the second line is simply an amalgamation of Equation and Equation. Complex matrices have finally entered the picture. We shall devote all of Chapter 10 to uncovering the remarkable properties enjoyed by the matrices that appear in the partial fraction expansion of $(zI - B)^{-1}$ Have you noticed that, in our example, the two matrices are each projections, and they sum to I. and that their product is 0? Could this be an accident?

In The Laplace Transform module we were confronted with the complex exponential. By analogy to the real exponential we define

$$e^z \equiv \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

and find that

$$\begin{aligned}e^e &= 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots \\ &= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4} - \cdots + i(\theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \cdots) \\ &= \cos\theta + i\sin\theta\end{aligned}$$

With this observation, the polar form is now simply $z = |z|e^{i\theta}$

One may just as easily verify that

$$\cos(\theta) = \frac{e^{i\theta} + e^{(-i)\theta}}{2}$$

and

$$\sin(\theta) = \frac{e^{i\theta} - e^{(-i)\theta}}{2i}$$

These suggest the definitions, for complex z

$$\cos(z) \equiv \frac{e^{iz} + e^{(-i)z}}{2}$$

and

$$\sin(z) \equiv \frac{e^{iz} - e^{(-i)z}}{2i}$$

As in the real case the exponential enjoys the property that

$$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

and in particular

$$\begin{aligned} e^{x+iy} &= e^x e^{iy} \\ &= e^x \cos(y) + i e^x \sin(y) \end{aligned}$$

Finally, the inverse of the complex exponential is the complex logarithm,

$$\ln(z) \equiv \ln(|z|) + i\theta$$

for $z = |z| e^{i\theta}$. One finds that $\ln(-1+i) = \ln(\sqrt{2}) + i\frac{3\pi}{4}$.

6.3: Complex Differentiation

The complex f is said to be differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, by which we mean that

$$\frac{f(z_n) - f(z_0)}{z_n - z_0}$$

converges to the same value **for every** sequence $\{z_n\}$ that converges to z_0. In this case we naturally call the limit $\frac{d}{dz} f(z_0)$

To illustrate the concept of '**for every**' mentioned above, we utilize the following picture. We assume the point z_0 is differentiable, which means that any conceivable sequence is going to converge to z_0. We outline three sequences in the picture: real numbers, imaginary numbers, and a spiral pattern of both.

Sequences Approaching A Point In The Complex Plane

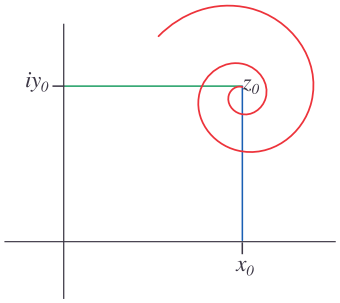


Figure 6.3.1: The green is real, the blue is imaginary, and the red is the spiral. (CC BY-NC; Ümit Kaya)

Example 6.3.1

The derivative of z^2 is $2z$.

$$\begin{aligned} \lim z \to z_0 \frac{z^2 - z_0^2}{z - z_0} &= \lim_{z \to z_0} \frac{(z - z_0)(z + z_0)}{z - z_0} \\ &= 2z_0 \end{aligned}$$

Example 6.3.2

The exponential is its own derivative.

$$\begin{aligned} \lim z \to z_0 \frac{e^z - e^{z_0}}{z - z_0} &= e^{z_0} \lim_{z \to z_0} \frac{e^{z - z_0} - 1}{z - z_0} \\ &= e^{z_0} \lim_{z \to z_0} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(n+1)!} \\ &= e^{z_0} \end{aligned}$$

Example 6.3.3

The real part of z is **not** a differentiable function of z.

We show that the limit depends on the angle of approach. First, when $z_n \to z_0$ on a line parallel to the real axis, e.g., $z_n = x_0 + \frac{1}{n} + iy_0$, we find

$$\lim_{n\to\infty} \frac{x_0+\frac{1}{n}-x_0}{x_0+\frac{1}{n}+iy_0-x_0+iy_0} = 1$$

while if $z_n \to z_0$ in the imaginary direction, e.g., $z_n = x_0 + i(y_0+\frac{1}{n})$, then

$$\lim_{n\to\infty} \frac{x_0-x_0}{x_0+i(y_0+\frac{1}{n})-x_0+iy_0} = 0$$

Conclusion

NOT_CONVERTED_YET: para

This last example suggests that when f is differentiable a simple relationship must bind its partial derivatives in x and y.

Definition: Partial Derivative Relationship

If f is differentiable at z_0 then $\frac{d}{dz}f(z_0) = \frac{\partial f(z_0)}{\partial x} = -(i\frac{\partial f(z_0)}{\partial y})$

With $z = x+iy_0$

$$\begin{aligned}\frac{d}{dz}f(z_0) &= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\\ &= \lim_{x\to x_0}\frac{f(x+iy_0)-f(x_0+iy_0}{x-x_0}\\ &= \frac{\partial f(z_0)}{\partial x}\end{aligned}$$

With $z = x_0+iy$

$$\begin{aligned}\frac{d}{dz}f(z_0) &= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\\ &= \lim_{x\to x_0}\frac{f(x_0+iy)-f(x_0+iy_0}{i(y-y_0)}\\ &= -(i\frac{\partial f(z_0)}{\partial y})\end{aligned}$$

Cauchy-Reimann Equations

In terms of the real and imaginary parts of f this result brings the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Regarding the converse proposition we note that when f has continuous partial derivatives in region obeying the Cauchy-Reimann equations then f is in fact differentiable in the region.

We remark that with no more energy than that expended on their real cousins one may uncover the rules for differentiating complex sums, products, quotients, and compositions.

As one important application of the derivative let us attempt to expand in partial fractions a rational function whose denominator has a root with degree larger than one. As a warm-up let us try to find $q_{1,1}$ and $q_{1,2}$ in the expression

$$\frac{z+2}{(z+1)^2} = \frac{q_{1,1}}{z+1} + \frac{q_{1,2}}{(z+1)^2}$$

Arguing as above, it seems wise to multiply through by $(z+1)^2$ and so arrive at

$$z+2=q_{1,1}(z+1)+q_{1,2}$$

On setting $z=-1$ this gives $q_{1,2}=1$. With $q_{1,2}$ computed, Equation takes the simple form $z+1=q_{1,1}(z+1)$ and so $q_{1,2}=1$ as well. Hence,

$$\frac{z+2}{(z+1)^2}=\frac{1}{z+1}\,\frac{1}{(z+1)^2}$$

This latter step grows more cumbersome for roots of higher degrees. Let us consider

$$\frac{(z+2)^2}{(z+1)^3}=\frac{q_{1,1}}{z+1}+\frac{q_{1,2}}{(z+1)^2}+\frac{q_{1,3}}{(z+1)^3}$$

The first step is still correct: multiply through by the factor at its highest degree, here 3. This leaves us with

$$(z+2)^2=q_{1,1}(z+1)^2+q_{1,2}(z+1)+q_{1,3}$$

Setting $z=-1$ again produces the last coefficient, here $q_{1,3}=1$. We are left however with one equation in two unknowns. Well, not really one equation, for Equation is to hold for **all** z, of Equation. This produces

$$2(z+2)=2q_{1,1}(z+1)+q_{1,2}$$

and $2=q_{1,1}$ The latter of course needs no comment. We derive $q_{1,2}$ from the former by setting $z=-1$. This example will permit us to derive a simple expression for the **partial fraction expansion** of the general proper rational function $q=\frac{f}{g}$ where g has h distinct roots $\{\lambda_1,\cdots,\lambda_h\}$ of respective degrees $\{d_1,\cdots,d_h\}$. We write

$$q(z)=\sum_{j=1}^{h}\sum_{k=1}^{d_j}\frac{q_{j,k}}{(z-\lambda_j)^k}$$

and note, as above, that $q_{j,k}$ is the coefficient of $(z-d_j)^{d_{j-k}}$ in the rational function

$$r_j(z)\equiv q(z)(z-\lambda_j)^{d_j}$$

Hence, $q_{j,k}$ may be computed by setting $z=\lambda_j$ in the ratio of the d_j-kth derivative of r_j to $(d_j-k)!$

$$q_{j,k}=\lim_{z\to\lambda_j}\frac{1}{(d_j-k)!}\frac{d^{d_j-k}}{dz^{d_j-k}}\{(z-\lambda_j)^{d_j}q(z)\}$$

As a second example, let us take

$$B=\begin{pmatrix}1&0&0\\1&3&0\\0&1&1\end{pmatrix}$$

and compute the $\Phi_{j,k}$ matrices in the expansion

$$\begin{aligned}(zI-B)^{-1}&=\begin{pmatrix}\frac{1}{z-1}&0&0\\\frac{1}{(z-1)(z-3)}&\frac{1}{z-3}&0\\\frac{1}{(z-1)^2(z-3)}&\frac{1}{(z-1)(z-3)}&\frac{1}{z-1}\end{pmatrix}\\&=\frac{1}{z-1}\Phi_{1,1}+\frac{1}{(z-1)^2}\Phi_{1,2}+\frac{1}{z-3}\Phi_{2,1}\end{aligned}$$

The only challenging term is the $(3,1)$ element. We write

$$\frac{1}{(z-1)^2(z-3)}=\frac{q_{1,1}}{z-1}+\frac{q_{1,2}}{(z-1)^2}+\frac{q_{2,1}}{z-3}$$

It follows that

$$q_{1,1} = \frac{d}{dz}\left(\frac{1}{z-3}1\right)$$
$$= -1/4$$

and

$$q_{1,2} = \frac{1}{z-3}1$$
$$= -1/4$$

and

$$q_{2,1} = \left(\frac{1}{(z-3)^2}1\right)$$
$$= 1/4$$

It now follows that

$$(zI-B)^{-1} = \frac{1}{z-1}\begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 0 \\ -1/4 & -1/2 & 1 \end{pmatrix} + \frac{1}{(z-1)^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix} + \frac{1}{z-3}\begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 1/2 & 0 \end{pmatrix}$$

In closing, let us remark that the method of partial fraction expansions has been implemented in Matlab. In fact, Equations $q_{1,1}, q_{1,2}, q_{2,1}$ all follow from the single command: `[r,p,k]=residue([0 0 0 1],[1 -5 7 -3])` . The first input argument is Matlab-speak for the polynomial $f(z) = 1$ while the second argument corresponds to the denominator

$$g(z) = (z-1)^2(z-3) = z^3 - 5z^2 + 7z - 3$$

6.4: Exercises- Complex Numbers, Vectors, and Functions

Exercise 6.4.1

Express $|e^z|$ in terms of x and/or y.

Exercise 6.4.2

Confirm that $e^{ln(z)} = z$ and $ln(e^z) = z$

Exercise 6.4.3

Find the real and imaginary parts of $\cos(z)$ and $\sin(z)$

Exercise 6.4.4

Show that $\cos^2(z) + \sin^2(z) = 1$

Exercise 6.4.5

With $z^w \equiv e^{wln(z)}$ for complex z and w compute $\sqrt{i}$

Exercise 6.4.6

Verify that $\cos(z)$ and $\sin(z)$ satisfy the Cauchy-Riemann equations and use the proposition to evaluate their derivatives.

Exercise 6.4.7

Submit a Matlab diary documenting your use of residue in the partial fraction expansion of the transfer function of

$$B = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

CHAPTER OVERVIEW

7: Complex Analysis II

7.1: Cauchy's Theorem

Introduction

Our main goal is a better understanding of the partial fraction expansion of a given transfer function. With respect to the example that closed the discussion of complex differentiation, see the equation - In this equation, we found

$$(zI-B)^{-1}=\frac{1}{z-\lambda_1}P_1+\frac{1}{(z-\lambda_1)^2}D_1+\frac{1}{z-\lambda_2}P_2$$

where the P_j and D_j enjoy the amazing properties

$$\begin{aligned}BP_1&=P_1B\\&=\lambda_1P_1+D_1\end{aligned}$$

and

$$BP_2=P_2B=\lambda_2P_2$$

$$P_1+P_2=I$$

$$P_1^2=P_1$$

$$P_2^2=P_2$$

and

$$D_1^2=0$$

$$\begin{aligned}P_1D_1&=D_1P_1\\&=D_1\end{aligned}$$

and

$$P_2D_1=D_1P_2=0 \tag{7.1.1}$$

In order to show that this **always** happens, i.e., that it is not a quirk produced by the particular B, we require a few additional tools from the theory of complex variables. In particular, we need the fact that partial fraction expansions may be carried out through complex integration.

Integration of Complex Functions Over Complex Curves

We shall be integrating complex functions over complex curves. Such a curve is parameterized by one complex valued or, equivalently, two real valued, function(s) of a real parameter (typically denoted by t). More precisely,

$$C\equiv\{z(t)=x(t)+iy(t)|a\le t\le b\}$$

For example, if $x(t)=y(t)=t$ while $a=0$ and $b=1$, then C is the line segment joining $0+i0$ to $1+i$.

We now define

$$\int f(z)dz=\equiv\int_a^b f(z(t))z'(t)dt$$

For example, if $C=\{t+it|0\le t\le 1\}$ as above and $f(z)=z$ then

$$\int zdz=\int_0^1(t+it)(1+i)dt=\int_0^1 t-t+i2tdt=i$$

while if C is the unit circle $\{e^{it}|0\le t\le 2\pi\}$ then

$$\int zdz=\in_0^{2\pi}e^{it}ie^{it}dt=i\int_0^{2\pi}e^{i2t}dt=i\int_0^{2\pi}\cos(2t)+i\sin(2t)dt=0$$

Remaining with the unit circle but now integrating $f(z)=\frac{1}{z}$ we find

$$\int z^{-1}dz=\int_0^{2\pi}e^{-(it)}ie^{it}dt=2\pi i$$

We generalize this calculation to arbitrary (integer) powers over arbitrary circles. More precisely, for integer mm and fixed complex $(z-a)^m$ over

$$C(a,r)\equiv\{a+re^{it}|0\leq t\leq 2\pi\}$$

the circle of radius r centered at a

$$\int(z-a)^m dz=\int_0^{2\pi}(a+re^{it}-a)^m rie^{it}dt$$

$$=ir^{m+1}\int_0^{2\pi}e^{i(m+1)t}dt$$

$$\int(z-a)^m dz=ir^{m+1}\int_0^{2\pi}\cos((m+1)t)+i\sin((m+1)t)dt=\begin{cases}2\pi i & \text{if} m=-1\\ 0 & \text{otherwise}\end{cases}$$

When integrating more general functions it is often convenient to express the integral in terms of its real and imaginary parts. More precisely

$$\begin{aligned}\int f(z)dz &= \int u(x,y)+iv(x,y)dx+i\int u(x,y)+iv(x,y)dy\\ &=\int u(x,y)dx-\int v(x,y)dy+i\int v(x,y)dx+i\int u(x,y)dy\\ &=\int_a^b u(x(t),y(t))x'(t)-v(x(t),y(t))y'(t)dt+i\int_a^b u(x(t),y(t))y'(t)+v(x(t),y(t))x'(t)dt\end{aligned}$$

The second line should invoke memories of:

Green's Theorem

If C is a closed curve and M and N are continuously differentiable real-valued functions on C_{in}, the region enclosed by C, then

$$\int Mdx+\int Ndy=\iint\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}dxdy$$

Applying this to the situation above, we find, so long as C is closed, that

$$\int f(z)dz=-\iint\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}dxdy+i\iint\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}dxdy$$

At first glance it appears that Green's Theorem only serves to muddy the waters. Recalling the Cauchy-Riemann equations however we find that each of these double integrals is in fact identically zero! In brief, we have proven:

Cauchy's Theorem

If f is differentiable on and in the closed curve C then $\int f(z)dz=0$.

Strictly speaking, in order to invoke Green's Theorem we require not only that ff be differentiable but that its derivative in fact be continuous. This however is simply a limitation of our simple mode of proof; Cauchy's Theorem is true as stated.

This theorem, together with $C(a,r)\equiv\{a+re^{it}|0\leq t\leq 2\pi\}$, permits us to integrate every proper rational function. More precisely, if $q=\frac{f}{g}$ where f is a polynomial of degree at most $m-1$ and g is an mth degree polynomial with h distinct zeros at $\{\lambda_j|j=\{1,\cdots,h\}\}$ with respective multiplicities of $\{m_j|j=\{1,\cdots,h\}\}$ we found that

$$q(z) = \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{q_{j,k}}{(z-\lambda_j)^k}$$

Observe now that if we choose r_j so small that λ_j is the only zero of g encircled by $C_j \equiv C(\lambda_j, r_j)$ then by Cauchy's Theorem

$$\int q(z)dz = \sum_{k=1}^{m_j} q_{j,k} \int \frac{1}{(z-\lambda_j)^k} dz$$

In Equation we found that each, save the first, of the integrals under the sum is in fact zero. Hence,

$$\int q(z)dz = 2\pi i q_{j,1}$$

With $q_{j,1}$ in hand, say from this equation or `residue`, one may view Equation as a means for computing the indicated integral. The opposite reading, i.e., that the integral is a convenient means of expressing $q_{j,1}$, will prove just as useful. With that in mind, we note that the remaining residues may be computed as integrals of the product of q and the appropriate factor. More precisely,

$$\int q(z)(z-\lambda_j)^{k-1} dz = 2\pi i q_{j,k}$$

One may be led to believe that the precision of this result is due to the very special choice of curve and function. We shall see ...

7.2: Cauchy's Integral Formula

The Residue Theorem

After Cauchy's Theorem perhaps the most useful consequence of Cauchy's Theorem is the The Curve Replacement Lemma

Suppose that C_2 is a closed curve that lies inside the region encircled by the closed curve C_1. If f is differentiable in the annular region outside C_2 and inside C_1 then

$$\int f(z)dz = \int f(z)dz$$

With reference to Figure 1, we introduce two vertical segments and define the closed curves $C_3 = abcda$ (where the bc arc is clockwise and the da arc is counterclockwise) and $C_4 = adcba$ (where the ad arc is counter-clockwise and the cb arc is clockwise). By merely following the arrows we learn that

$$\int f(z)dz = \int f(z)dz + \int f(z)dz + \int f(z)dz$$

As Cauchy's Theorem implies that the integrals over C_3 and C_4 each vanish, we have our result.

Curve Replacement Figure

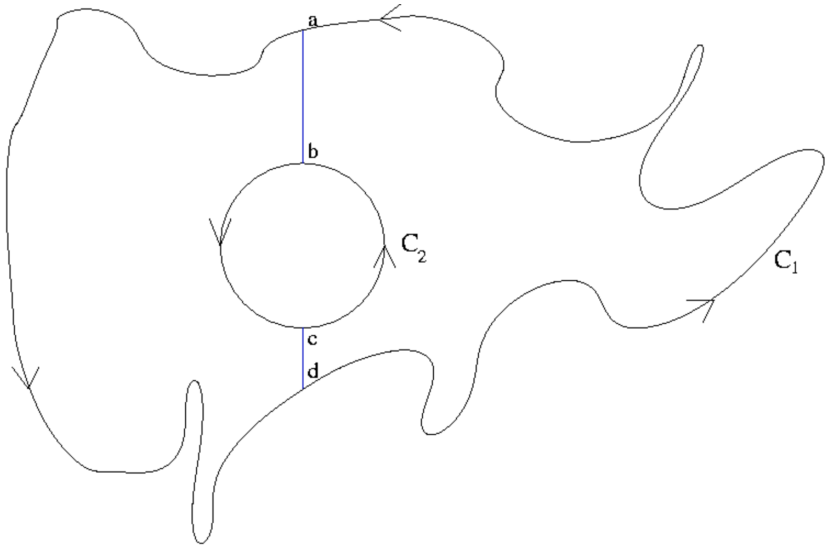


Figure 1. The Curve Replacement Lemma

This Lemma says that in order to integrate a function it suffices to integrate it over regions where it is singular, i.e. nondifferentiable.

Let us apply this reasoning to the integral

$$\int \frac{z}{(z-\lambda_1)(z-\lambda_2)}dz$$

where C encircles both λ_1 and λ_2 as depicted in Figure. We find that

$$\int \frac{z}{(z-\lambda_1)(z-\lambda_2)}dz = \int \frac{z}{(z-\lambda_1)(z-\lambda_2)}dz + \int \frac{z}{(z-\lambda_1)(z-\lambda_2)}dz$$

Developing the integrand in partial fractions we find

$$\int \frac{z}{(z-\lambda_1)(z-\lambda_2)}dz = \int \frac{\lambda_1}{\lambda_1-\lambda_2}\int \frac{1}{(z-\lambda_1)}dz + \int \frac{\lambda_2}{\lambda_2-\lambda_1}\int \frac{1}{(z-\lambda_2)}dz$$

$$= \frac{2\pi i\lambda_1}{\lambda_1-\lambda_2}$$

Similarly,

$$\int \frac{z}{(z-\lambda_1)(z-\lambda_2)}dz = \frac{2\pi i \lambda_1}{\lambda_1 - \lambda_2}$$

Putting things back together we find

$$\begin{aligned}\int \frac{z}{(z-\lambda_1)(z-\lambda_2)}dz &= 2\pi i \left(\frac{\lambda_1}{\lambda_1 - \lambda_2} - \frac{\lambda_2}{\lambda_2 - \lambda_1} \right) \\ &= 2\pi i\end{aligned}$$

Figure 2. Concentrating on the poles.

We may view Equation as a special instance of integrating a rational function around a curve that encircles all of the zeros of its denominator. In particular, recalling that Cauchy's Theorem, we find

$$\int q(z)dz = \sum_{j=1}^{h} \sum_{k=1}^{m_j} \int \frac{q_{j,k}}{(z-\lambda_j)^k}dz = 2\pi i \sum_{j=1}^{h} q_{j,1}$$

To take a slightly more complicated example let us integrate $\frac{f(z)}{z-a}$ over some closed curve C inside of which f is differentiable and a resides. Our Curve Replacement Lemma now permits us to claim that

$$\int \frac{f(z)}{z-a}dz = \int \frac{f(z)}{z-a}dz$$

It appears that one can go no further without specifying f. The alert reader however recognizes that in the integral over $C(a,r)$ is independent of rr and so proceeds to let $r \to 0$, in which case $z \to a$ and $f(z) \to f(a)$. Computing the integral of $\frac{1}{z-a}$ along the way we are led to the hope that

$$\int \frac{f(z)}{z-a}dz = f(a)2\pi i$$

In support of this conclusion we note that

$$\int \frac{f(z)}{z-a}dz = \int \frac{f(z)}{z-a}dz + \int \frac{f(a)}{z-a}dz - \int \frac{f(a)}{z-a}dz f(a) = f(a)\int \frac{1}{z-a}dz + \int \frac{f(z)-f(a)}{z-a}dz$$

Now the first term is $f(a)2\pi i$ regardless of r while, as $r \to 0$ the integrand of the second term approaches $\frac{d}{da}f(a)$ and the region of integration approaches the point a. Regarding this second term, as the integrand remains bounded as the perimeter of $C(a,r)$ approaches zero the value of the integral must itself be zero. This result if typically known as

Cauchy's Integral Equation

If f is differentiable on and in the closed curve C then

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}dz$$

for each a lying inside C.

The consequences of such a formula run far and deep. We shall delve into only one or two. First, we note that, as a does not lie on C, the right hand side is a perfectly smooth function of a. Hence, differentiating each side, we find

$$\frac{d}{da}f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^2}dz$$

for each a lying inside C. Applying this reasoning nn times we arrive at a formula for the n-th derivative of f at a

$$\frac{d^n}{da^n}f(a) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{1+n}}dz$$

for each a lying inside C. The upshot is that once f is shown to be differentiable it must in fact be infinitely differentiable. As a simple extension let us consider

$$\frac{1}{2\pi i}\int \frac{f(z)}{(z-\lambda_1)(z-\lambda_2)^2}dz$$

where f is still assumed differentiable on and in C and that C encircles both λ_1 and λ_2. By the curve replacement lemma this integral is the sum

$$\frac{1}{2\pi i}\int \frac{f(z)}{(z-\lambda_1)(z-\lambda_2)^2}dz + \frac{1}{2\pi i}\int \frac{f(z)}{(z-\lambda_1)(z-\lambda_2)^2}$$

where λ_j now lies in only C_j. As $\frac{f(z)}{z-\lambda_2}$ is well behaved in C_1 we may use Equation to conclude that

$$\frac{1}{2\pi i}\int \frac{f(z)}{(z-\lambda_1)(z-\lambda_2)^2}dz = \frac{f(\lambda_1)}{(\lambda_1-\lambda_2)^2}$$

Similarly, as $\frac{f(z)}{z-\lambda_1}$ is well behaved in C_2 we may use Equation to conclude that

$$\frac{1}{2\pi i}\int \frac{f(z)}{(z-\lambda_1)(z-\lambda_2)^2}dz = \frac{d}{da}(\frac{f(a)}{a-\lambda_1})|_{a=\lambda_2}$$

These calculations can be read as a concrete instance of

The Residue Theorem

If g is a polynomial with roots $\{\lambda_j | j = \{1, \cdots, h\}\}$ of degree $\{d_j | j = \{1, \cdots, h\}\}$ and C is a closed curve encircling each of the λ_j and f is differentiable on and in C then

$$\int \frac{f(z)}{g(z)}dz = 2\pi i \sum_{j=1}^{h} res(\lambda_j)$$

where

$$res(\lambda_j) = lim_{z\to\lambda_j}\frac{1}{(d_j-1)!}\frac{d^{d_j-1}(z-\lambda_j)^{d_j}\frac{f(z)}{g(z)}}{dz^{d_j-1}}$$

is called **the residue of $\frac{f}{g}$ at λ_j**.

One of the most important **instances** of this theorem is the formula for the Inverse Laplace Transform.

7.3: The Inverse Laplace Transform- Complex Integration

The Inverse Laplace Transform

If q is a rational function with poles $\{\lambda_j | j = \{1, \cdots, h\}\}$ then the inverse Laplace transform of q is

$$\mathscr{L}^{-1}(q)(t) \equiv \frac{1}{2\pi i}\int q(z)e^{zt}dz$$

where C is a curve that encloses each of the poles of q

$$\mathscr{L}^{-1}(q)(t) = \sum_{j=1}^{h} res(\lambda_j)$$

Let us put this lovely formula to the test. We take our examples from discussion of the Laplace Transform and the inverse Laplace Transform. Let us first compute the inverse Laplace Transform of

$$q(z) = \frac{1}{(z+1)^2}$$

According to Equation it is simply the residue of $q(z)e^{zt}$ at $z = -1$ i.e.,

$$res(-1) = \lim_{z \to -1} de^{zt}dz = te^{-t}$$

This closes the circle on the example begun in the discussion of the Laplace Transform and continued in exercise one for chapter 6.

For our next example we recall

$$\mathscr{L}(x_1(s)) = \frac{0.19(s^2 + 1.5s + 0.27)}{(s+1/6)^4(s^3 + 1.655s^2 + 0.4978s + 0.0039)}$$

from the Inverse Laplace Transform. Using `numde` , `sym2poly` and `residue` , see `fib4.m` for details, returns

$$r_1 = \begin{pmatrix} 0.0029 \\ 262.8394 \\ -474.1929 \\ -1.0857 \\ -9.0930 \\ -0.3326 \\ 211.3507 \end{pmatrix}$$

and

$$p_1 = \begin{pmatrix} -1.3565 \\ -0.2885 \\ -0.1667 \\ -0.1667 \\ -0.1667 \\ -0.1667 \\ -0.0100 \end{pmatrix}$$

You will be asked in the exercises to show that this indeed jibes with the

$$x_1(t) = 211.35e^{\frac{-t}{100}} - (0.0554t^3 + 4.5464t^2 + 1.085t + 474.19)e^{\frac{-t}{6}}$$
$$+ e^{\frac{-329t}{400}}(262.842\cosh(\frac{\sqrt{73}t}{16}) + 262.836\sinh(\frac{\sqrt{73}t}{16}))$$

achieved in the Laplace Transform via `ilaplace` .

7.4: Exercises- Complex Integration

Exercise 7.4.1

Let us confirm the representation of this Cauchy's Theorem equation in the matrix case. More precisely, if $\Phi(z) \equiv (zI - B)^{-1}$ is the transfer function associated with B then this Cauchy's Theorem equation states that

$$\Phi(z) = \sum_{j=1}^{h} \sum_{k=1}^{d_j} \frac{\Phi_{j,k}}{(z-\lambda_j)^k}$$

where

$$\Phi_{j,k} = \frac{1}{2\pi i} \int \frac{\Phi(z)}{(z-\lambda_j)^{k-1}} dz$$

Compute the $\Phi_{j,k}$ per Equation for the B in this equation from the discussion of Complex Differentiation. Confirm that they agree with those appearing in this equation from the Complex Differentiation discussion.

Exercise 7.4.2

Use the inverse Laplace Transform equation to compute the inverse Laplace transform of $\frac{1}{s^2+2s+2}$.

Exercise 7.4.3

Use the result of the previous exercise to solve, via the Laplace transform, the differential equation

$$\frac{d}{dt}(x)(t) + x(t) = e^{-t\sin(t)}, \quad x(0) = 0$$

Hint: Take the Laplace transform of each side.

Exercise 7.4.4

Explain how one gets from r_1 and p_1 to $x_1(t)$.

Exercise 7.4.5

Compute, as in `fib4.m` , the residues of $\mathscr{L}(x_2(s))$ and $\mathscr{L}(x_3(s))$ and confirm that they give rise to the $x_2(t)$ and $x_3(t)$ you derived in the discussion of Chapter 1.1.

CHAPTER OVERVIEW

8: The Eigenvalue Problem

8.1: Introduction to the Eigenvalue Problem

Introduction

Harking back to our previous discussion of The Laplace Transform we labeled the complex number λ an **eigenvalue** of B if $\lambda I - B$ was not invertible. In order to find such λ one has only to find those s for which $(sI-B)^{-1}$ is not defined. To take a concrete example we note that if

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

then

$$(sI-B)^{-1} = \frac{1}{(s-1)^2(s-2)} \begin{pmatrix} (s-1)(s-2) & s-2 & 0 \\ 0 & (s-1)(s-2) & 0 \\ 0 & 0 & (s-1)^2 \end{pmatrix}$$

and so $\lambda_1 = 1$ and $\lambda_2 = 2$ are the two eigenvalues of B. Now, to say that $\lambda_j I - B$ is not invertible is to say that its columns are linearly dependent, or, equivalently, that the null space $\mathcal{N}(\lambda_j I - B)$ contains more than just the zero vector. We call $\mathcal{N}(\lambda_j I - B)$ the jth **eigenspace** and call each of its nonzero members a jth **eigenvector**. The dimension of $\mathcal{N}(\lambda_j I - B)$ is referred to as the **geometric multiplicity** of λ_j. With respect to B above, we compute $\mathcal{N}(\lambda_j I - B)$ by solving $(I-B)x = 0$ i.e.,

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Clearly,

$$\mathcal{N}(\lambda_1 I - B) = \{c(1 \quad 0 \quad 0)^T | c \in \mathbb{R}\}$$

Arguing along the same lines we also find

$$\mathcal{N}(\lambda_2 I - B) = \{c(0 \quad 0 \quad 1)^T | c \in \mathbb{R}\}$$

That B is 3×3 but possesses only 2 linearly eigenvectors leads us to speak of B as defective. The cause of its defect is most likely the fact that λ_1 is a double pole of $(sI-B)^{-1}$. In order to flesh out that remark and uncover the missing eigenvector we must take a much closer look at the transfer function

$$R(s) \equiv (sI-B)^{-1}$$

In the mathematical literature this quantity is typically referred to as the **Resolvent** of B.

8.2: The Resolvent

The Transfer Function

One means by which to come to grips with $R(s)$ is to treat it as the matrix analog of the scalar function

$$\frac{1}{s-b}$$

This function is a scaled version of the even simpler function $\frac{1}{1-z}$ This latter function satisfies the identity (just multiply across by $1-z$ to check it)

$$\frac{1}{1-z} = 1+z+z^2+\cdots+z^{n-1}+\frac{z^n}{1-z}$$

for each positive integer n. Furthermore, if $|z|<1$ then $z^n \to 0$ as $n \to \infty$ and so Equation becomes, in the limit,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

the familiar geometric series. Returning to Equation we write

$$\frac{1}{s-b} = \frac{\frac{1}{s}}{1-\frac{b}{s}} = \frac{1}{s}+\frac{b}{s^2}+\cdots+\frac{b^{n-1}}{s^n}+\frac{b^n}{s^n}\frac{1}{s-b}$$

and hence, so long as $|s|>|b|$ we find,

$$\frac{1}{s-b} = \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{b}{s}\right)^n$$

This same line of reasoning may be applied in the matrix case. That is,

$$(sI-B)^{-1} = s^{-1}\left(I-\frac{B}{s}\right)^{-1}\left(\frac{1}{s}+\frac{B}{s^2}+\cdots+\frac{B^{n-1}}{s^n}+\frac{B^n}{s^n}(sI-B)^{-1}\right)$$

and hence, so long as $|s|>||B||$ where $||B||$ is the magnitude of the largest element of B

$$\frac{1}{sI-B} = s^{-1}\sum_{n=0}^{\infty}\left(\frac{B}{s}\right)^n$$

Although Equation is indeed a formula for the transfer function you may, regarding computation, not find it any more attractive than the Gauss-Jordan method. We view Equation however as an analytical rather than computational tool. More precisely, it facilitates the computation of integrals of $R(s)$. However, C_ρ is the circle of radius ρ centered at the origin and $\rho>||B||$ then

$$\begin{aligned}\int_{C_\rho}(sI-B)^{-1}ds &= \sum_{n=0}^{\infty}B^n\int_{C_\rho}s^{-1-n}ds \\ &= 2\pi i I\end{aligned}$$

This result is essential to our study of the eigenvalue problem. As are the two resolvent identities. Regarding the first we deduce from the simple observation

$$(s_2I-B)^{-1}-(s_1I-B)^{-1} = (s_2I-B)^{-1}(s_1I-B-s_2I+B)(s_1I-B)^{-1}$$

that

$$R(s_2)-R(s_1) = (s_1-s_2)R(s_2)R(s_1)$$

The second identity is simply a rewriting of

$$(sI-B)(sI-B)^{-1} = (sI-B)^{-1}(sI-B) = I$$

namely,

$$\begin{aligned} BR(s) &= R(s)B \\ &= sR(s) - I \end{aligned}$$

8.3: The Partial Fraction Expansion of the Resolvent

Partial Fraction Expansion of the Transfer Function

The Gauss-Jordan method informs us that R will be a matrix of rational functions with a common denominator. In keeping with the notation of the previous chapters, we assume the denominator to have the h distinct roots, $\{\lambda_j | j = \{1, \cdots, h\}\}$ with associated multiplicities $\{m_j | j = \{1, \cdots, h\}\}$

Now, assembling the partial fraction expansions of each element of R we arrive at

$$R(s) = \sum_{j=1}^{h} \sum_{k=1}^{m_j} \frac{R_{j,k}}{(s-\lambda_j)^k}$$

where, recalling the equation from Cauchy's Theorem, the matrix $R_{j,k}$ equals the following:

$$R_{j,k} = \frac{1}{2\pi j} \int R(z)(z-\lambda_j)^{k-1} dz$$

Example 8.3.1

As we look at this example in the introduction, we find

$$R_{1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad R_{1,1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad R_{2,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One notes immediately that these matrices enjoy some amazing properties. For example

$$R_{1,1}^2 = R_{1,1} \quad R_{1,2}^2 = R_{1,2} \quad R_{1,1}R_{2,1} = 0 \quad \textit{and} \quad R_{2,1}^2 = R_{2,1}$$

Below we will now show that this is no accident. As a consequence of Equation and the first resolvent identity, we shall find that these results are true in general.

$R_{j,1}^2 = R_{j,1}$ as seen above.

Recall that the C_j appearing in Equation is any circle about λ_j that neither touches nor encircles any other root. Suppose that C_j and C_j' are two such circles and C_j' encloses C_j. Now,

$$R_{j,1} = \frac{1}{2\pi j} \int R(z) dz = \frac{1}{2\pi j} \int R(z) dz$$

and so

$$R_{j,1}^2 = \frac{1}{(2\pi j)^2} \int R(z) dz = \frac{1}{2\pi j} \int R(w) dw$$

$$R_{j,1}^2 = \frac{1}{(2\pi j)^2} \int \int R(z) R(w) dw dz$$

$$R_{j,1}^2 = \frac{1}{(2\pi j)^2} \int \int \frac{R(z) - R(w)}{w - z} dw dz$$

$$R_{j,1}^2 = \frac{1}{(2\pi j)^2} (\int R(z) - \int \frac{1}{w-z} dw dz - \int R(w) - \int \frac{1}{w-z} dz dw)$$

$$R_{j,1}^2 = \frac{1}{2\pi i} \int R(z) dz = R_{j,1}$$

We used the first resolvent identity, This Transfer Function equation, in moving from the second to the third line. In moving from the fourth to the fifth we used only

$$\int \frac{1}{w-z} dw = 2\pi i$$

and

$$\int \frac{1}{w-z} dz = 0$$

The latter integrates to zero because C_j does not encircle ww

From the definition of orthogonal projections, which states that matrices that equal their squares are projections, we adopt the abbreviation

$$P_j \equiv R_{j,1}$$

With respect to the product $P_j P_k$, for $j \neq k$, the calculation runs along the same lines. The difference comes in Equation where, as C_j lies completely outside of C_k, both integrals are zero. Hence,

If $j \neq k$ then $P_j P_k = 0$

Along the same lines we define

$$D_j \equiv R_{j,2}$$

and prove

If $1 \leq k \leq m_j - 1$ then $D_j^k = R_{j,k+1}$ \cdot D_{j}^{m_{j}} = 0\)

For k and l greater than or equal to one,

$$R_{j,k+1} R_{j,l+1} = \frac{1}{(2\pi i)^2} \int R(z)(z-\lambda_j)^k dz \int R(w)(w-\lambda_j)^l dw$$

$$R_{j,k+1} R_{j,l+1} = \frac{1}{(2\pi i)^2} \int \int R(z) R(w)(z-\lambda_j)^k (w-\lambda_j)^l dw dz$$

$$R_{j,k+1} R_{j,l+1} = \frac{1}{(2\pi i)^2} \int \int \frac{R(z)-R(w)}{w-z} (z-\lambda_j)^k (w-\lambda_j)^l dw dz$$

$$R_{j,k+1} R_{j,l+1} = \frac{1}{(2\pi i)^2} \int R(z)(z-\lambda_j)^k \int \frac{(w-\lambda_j)^l}{w-z} dw dz - \frac{1}{(2\pi i)^2} \int R(w)(w-\lambda_j)^k \int \frac{(z-\lambda_j)^k}{w-z} dz dw$$

$$R_{j,k+1} R_{j,l+1} = \frac{1}{2\pi i} \int R(z)(z-\lambda_j)^{k+l} dz = R_{j,k+l+1}$$

because

$$\int \frac{(w-\lambda_j)^l}{w-z} dw = 2\pi i (z-\lambda_j)^l$$

and

$$\int \frac{(z-\lambda_j)^k}{w-z} dw = 0$$

With $k = l = 1$ we have shown $R_{j,2}^2 = R_{j,3}$ i.e., $D_j^2 = R_{j,3}$. Similarly, with $k = 1$ and $l = 2$ we find $R_{j,2} R_{j,3} = R_{j,4}$ i.e., $D_j^3 = R_{j,4}$. Continuing in this fashion we find $R_{j,k} R_{j,k+1} = R_{j,k+2} = j$, or $D_j^{k+1} = R_{j,k+2}$. Finally, at $k = m_{j-1}$ this becomes

$$D_j^{m_j} = R_{j,m_j+1} = \frac{1}{2\pi i} \int R(z)(z-\lambda_j)^{m_j} dz = 0$$

by Cauchy's Theorem.

With this we now have the sought after expansion

$$R(z) = \sum_{j=1}^{h} \frac{1}{z - \lambda_j} P_j + \sum_{k=1}^{m_{j-1}} \frac{1}{(z - \lambda_j)^{k+1}} D_j^k$$

along with the verification of a number of the properties laid out in Complex Integration Equations.

8.4: The Spectral Representation

With just a little bit more work we shall arrive at a similar expansion for BB itself. We begin by applying the second resolvent identity to P_j. More precisely, we note that the second resolvent identity implies that

$$BP_j = P_jB$$

$$= \int_{C_j} zR(z) - Idz$$

$$P_jB = \int_{C_j} zR(z)dz$$

$$P_jB = \int_{C_j} R(z)(z-\lambda_j)dz + \lambda_j \int_{C_j} R(z)dz$$

$$P_jB = D_j + \lambda_j P_j$$

Summing this over j we find

$$B\sum_{j=1}^{h} Pj = \sum_{j=1}^{h} \lambda_j P_j + \sum_{j=1}^{h} D_j$$

We can go one step further, namely the evaluation of the first sum. This stems from the equation in the discussion of the transfer function where we integrated $R(s)$ over a circle C_ρ where $\rho > ||B||$. The connection to the P_j is made by the residue theorem. More precisely,

$$\int_{C_\rho} R(z)dz = 2\pi i \sum_{j=1}^{h} P_j$$

Comparing this to the equation from the discussion of the transfer function we find

$$\sum_{j=1}^{h} P_j = I$$

and so takes the form

$$B = \sum_{j=1}^{h} \lambda_j P_j + \sum_{j=1}^{h} D_j$$

It is this formula that we refer to as the **Spectral Representation** of B. To the numerous connections between the P_j and D_j we wish to add one more. We first write as

$$(B - \lambda_j I)P_j = D_j$$

and then raise each side to the m_j power. As $P_j^{m_j} = P_j$ and $D_j^{m_j} = 0$ we find

$$(B - \lambda_j I)m_j^{P_j} = 0$$

For this reason we call the range of P_j the jth **generalized eigenspace**, call each of its nonzero members a jth **generalized eigenvector** and refer to the dimension of $\mathscr{R}(P_j)$ as the **algebraic multiplicity** of λ_j. With regard to the first example from the discussion of the eigenvalue problem, we note that although it has only two linearly independent eigenvectors the span of the associated generalized eigenspaces indeed fills out $\mathbb{R}^3$. One may view this as a consequence of $P_1 + P_2 = I$ or, perhaps more concretely, as appending the generalized first eigenvector $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$ to the original two eigenvectors $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$. In still other words, the algebraic multiplicities sum to the ambient dimension (here 3), while the sum of geometric multiplicities falls short (here 2).

8.5: The Eigenvalue Problem- Examples

We take a look back at our previous examples in light of the results of two previous sections The Spectral Representation and The Partial Fraction Expansion of the Transfer Function. With respect to the rotation matrix

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we recall, see Cauchy's Theorem, that

$$R(s) = \frac{1}{s^2+1}\begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}$$

$$R(s) = \frac{1}{s-i}\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix} + \frac{1}{s+i}\begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}$$

$$R(s) = \frac{1}{s-\lambda_1}P_1 + \frac{1}{s-\lambda_2}P_2$$

and so

$$B = \lambda_1 P_1 + \lambda_2 P_2 = i\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix} - i\begin{pmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{pmatrix}$$

From $m_1 = m_2 = 1$ it follows that $\mathscr{R}(P_1)$ and $\mathscr{R}(P_2)$ are actual (as opposed to generalized) eigenspaces. These column spaces are easily determined. In particular, $\mathscr{R}(P_1)$ is the span of

$$e_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

while $\mathscr{R}(P_2)$ is the span of

$$e_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

To recapitulate, from partial fraction expansion one can read off the projections from which one can read off the eigenvectors. The reverse direction, producing projections from eigenvectors, is equally worthwhile. We laid the groundwork for this step in the discussion of Least Squares. In particular, this Least Squares projection equation stipulates that

$$P_1 = e_1(e_1^T e_1)^{-1} e_1^T \quad and \quad P_2 = e_2(e_2^T e_2)^{-1} e_2^T$$

As $e_1^T e_1 = e_1^T e_1 = 0$ these formulas can not possibly be correct. Returning to the Least Squares discussion we realize that it was, perhaps implicitly, assumed that all quantities were real. At root is the notion of the length of a complex vector. It is not the square root of the sum of squares of its components but rather the square root of the sum of squares of the **magnitudes** of its components. That is, recalling that the magnitude of a complex quantity z is $\sqrt{z\overline{z}}$

$$(||e_1||)^2 \neq e_1^T e_1 \quad rather \quad (||e_1||)^2 \neq \overline{e_1}^T e_1$$

Yes, we have had this discussion before, recall complex numbers, vectors, and matrices. The upshot of all of this is that, when dealing with complex vectors and matrices, one should conjugate before every transpose. Matlab (of course) does this automatically, i.e., the ' symbol conjugates and transposes simultaneously. We use x^H to denote 'conjugate transpose', i.e.,

$$x^H \equiv \overline{x}^T$$

All this suggests that the desired projections are more likely

$$P_1 = e_1(e_1^H e_1)^{-1} e_1^H \quad and \quad P_2 = e_2(e_2^H e_2)^{-1} e_2^H$$

8.6: The Eigenvalue Problem- Exercises

Exercise 8.6.1

Argue as in Proposition 1 in the discussion of the partial fraction expansion of the transfer function that if $j \neq k$ then $D_j P_k = P_j D_k = 0$.

Exercise 8.6.2

Argue from the equation from the discussion of the Spectral Representation that $D_j P_j = P_j D_j = D_j$.

Exercise 8.6.3

The two previous exercises come in very handy when computing powers of matrices. For example, suppose B is 4-by-4, that $h = 2$ and $m_1 = m_2 = 2$. Use the spectral representation of B together with the first two exercises to arrive at simple formulas for B^2 and B^3.

Compute the spectral representation of the circulant matrix

$$B = \begin{pmatrix} 2 & 8 & 6 & 4 \\ 4 & 2 & 8 & 6 \\ 6 & 4 & 2 & 8 \\ 8 & 6 & 4 & 2 \end{pmatrix}$$

Carefully label all eigenvalues, eigenprojections and eigenvectors.

CHAPTER OVERVIEW

9: The Symmetric Eigenvalue Problem

9.1: The Spectral Representation of a Symmetric Matrix

Introduction

Our goal is to show that if B is symmetric then

- each λ_j is real,
- each P_j is symmetric and
- each D_j vanishes.

Let us begin with an example.

Example

The transfer function of

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

is

$$R(s) = \frac{1}{s(s-3)} \begin{pmatrix} s-2 & 1 & 1 \\ 1 & s-2 & 1 \\ 1 & 1 & s-2 \end{pmatrix}$$

$$R(s) = \frac{1}{s} \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & -1/3 \end{pmatrix} + \frac{1}{s-3} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$R(s) = \frac{1}{s-\lambda_1} P_1 + \frac{1}{s-\lambda_2} P_2$$

and so indeed each of the bullets holds true. With each of the D_j falling by the wayside you may also expect that the respective geometric and algebraic multiplicities coincide.

The Spectral Representation

We have amassed anecdotal evidence in support of the claim that each D_j in the spectral representation

$$B = \sum_{j=1}^{h} \lambda_j P_j + \sum_{j=1}^{h} D_j$$

is the zero matrix when B is symmetric, i.e., when $B = B^T$, or, more generally, when $B = B^H$ where $B^H \equiv \overline{B}^T$ Matrices for which $B = B^H$ are called **Hermitian**. Of course real symmetric matrices are Hermitian.

Taking the conjugate transpose throughout we find,

$$B^H = \sum_{j=1}^{h} \overline{\lambda_j} P_j^H + \sum_{j=1}^{h} D_j^H$$

That is, the $\overline{\lambda_j}$ are the eigenvalues of B^H with corresponding projections P_j^H and nilpotents D_j^H Hence, if $B = B^H$, we find on equating terms that

$$\lambda_j = \overline{\lambda_j}$$

$$P_j = P_j^H$$

and

$$D_j = D_j^H$$

The former states that the eigenvalues of an Hermitian matrix are real. Our main concern however is with the consequences of the latter. To wit, notice that for **arbitrary** x

$$(||D_j^{m_j-1}x||)^2 = x^H(D_j^{m_j-1})^H D_j^{m_j-1}x$$

$$(||D_j^{m_j-1}x||)^2 = x^H D_j^{m_j-1} D_j^{m_j-1}x$$

$$(||D_j^{m_j-1}x||)^2 = x^H D_j^{m_j-2} D_j^{m_j}x$$

$$(||D_j^{m_j-1}x||)^2 = 0$$

As $D_j^{m_j-1}x = 0$ for every x it follows (recall the previous exercise) that $D_j^{m_j-1} = 0$. Continuing in this fashion we find $D_j^{m_j-2} = 0$, and so, eventually, $D_j = 0$. If, in addition, B is real then as the eigenvalues are real and all the D_j vanish, the P_j must also be real. We have now established

If B is real and symmetric then

$$B = \sum_{j=1}^{h} \lambda_j P_j$$

where the λ_j are real and the P_j are real orthogonal projections that sum to the identity and whose pairwise products vanish.

One indication that things are simpler when using the spectral representation is

$$B^{100} = \sum_{j=1}^{h} \lambda_j^{100} P_j$$

As this holds for all powers it even holds for power series. As a result,

$$e^B = \sum_{j=1}^{h} e^{\lambda_j} P_j$$

It is also extremely useful in attempting to solve

$$Bx = b$$

for x. Replacing B by its spectral representation and b by Ib or, more to the point by $\sum_j P_j b$ we find

$$\sum_{j=1}^{h} \lambda_j P_j x = \sum_{j=1}^{h} P_j b$$

Multiplying through by P_1 gives $\lambda_1 P_1 x = P_1 b$ or $P_1 x = \frac{P_1 b}{\lambda_1}$. Multiplying through by the subsequent P_j's gives $P_j x = \frac{P_j b}{\lambda_j}$

Hence,

$$x = \sum_{j=1}^{h} P_j x$$

$$\sum_{j=1}^{h} \frac{1}{\lambda_j} P_j b$$

We clearly run in to trouble when one of the eigenvalues vanishes. This, of course, is to be expected. For a zero eigenvalue indicates a nontrivial null space which signifies dependencies in the columns of B and hence the lack of a unique solution to $Bx = b$.

Another way in which may be viewed is to note that, when B is symmetric, this previous equation takes the form

$$(zI - B)^{-1} = \sum_{j=1}^{h} \frac{1}{z - \lambda_j} P_j$$

Now if 0 is not an eigenvalue we may set $z = 0$ in the above and arrive at

$$B^{-1} = \sum_{j=1}^{h} \frac{1}{\lambda_j} P_j$$

Hence, the solution to $Bx = b$B is

$$x = B^{-1}b = \sum_{j=1}^{h} \frac{1}{\lambda_j} P_j b$$

We have finally reached a point where we can begin to define an inverse even for matrices with dependent columns, i.e., with a zero eigenvalue. We simply exclude the offending term in link. Supposing that $\lambda_h = 0$ we define the **pseudo-inverse** of B to be

$$B^{+} = \sum_{j=1}^{h-1} \frac{1}{\lambda_j} P_j$$

Let us now see whether it is deserving of its name. More precisely, when $b \in \mathscr{R}(B)$ we would expect that $x = B^{+}b$ indeed satisfies $Bx = b$. Well

$$BB^{+}b = B\sum_{j=1}^{h-1} \frac{1}{\lambda_j} P_j b = \sum_{j=1}^{h-1} \frac{1}{\lambda_j} BP_j b = \sum_{j=1}^{h-1} \frac{1}{\lambda_j} \lambda_j P_j b = \sum_{j=1}^{h-1} P_j b$$

It remains to argue that the latter sum really is b. We know that

$$\forall b, b \in \mathscr{R}(B) : (b = \sum_{j=1}^{h} P_j b)$$

The latter informs us that $b \perp N(B^T)$. As $B = B^T$, we have, in fact, that $b \perp N(B)$. AS P_h is nothing but orthogonal projection onto $N(B)$ it follows that $P_h b = 0$ and so $B(B^{+}b) = b$, that is, $x = B^{+}b$ is a solution to $Bx = b$ The representation is unarguably terse and in fact is often written out in terms of individual eigenvectors. Let us see how this is done. Note that if $x \in \mathscr{R}(P_1)$ then $x = P_1 x$ and so,

$$Bx = BP_1 x = \sum_{j=1}^{h} \lambda_j P_j P_1 x = \lambda_1 P_1 x = \lambda_1 x$$

i.e., x is an eigenvector of B associated with λ_1. Similarly, every (nonzero) vector $\mathscr{R}(P_j)$ is an eigenvector of B associated with λ_j.

Next let us demonstrate that each element of $\mathscr{R}(P_j)$ is orthogonal to each element of $\mathscr{R}(P_k)$ when $j \neq k$. If $x \in \mathscr{R}(P_j)$ and $x \in \mathscr{R}(P_k)$ then

$$x^T y = (P_j x)^T P_k y = x^T P_j P_k y = 0$$

With this we note that if $\{x_{j,1}, x_{j,2}, \cdots, x_{j,n_j}\}$constitutes a basis for $\mathscr{R}(P_j)$ then in fact the union of such bases,

$$\{x_{j,p} | (1 \leq j \leq h) \wedge (1 \leq p \leq n_j)\}$$

forms a linearly independent set. Notice now that this set has

$$\sum_{j=1}^{h} n_j$$

elements. That these dimensions indeed sum to the ambient dimension, n, follows directly from the fact that the underlying P_j sum to the $n - by - n$ identity matrix. We have just proven.

If B is real and symmetric and $n-by-n$, then B has a set of nn linearly independent eigenvectors.

Getting back to a more concrete version of link we now assemble matrices from the individual bases

$$E_j \equiv \{x_{j,1}, x_{j,2}, \cdots, x_{j,n_j}\}$$

and note, once again, that $P_j = E_j(E_j^T E_j)^{-1} E_j^T$, and so,

$$B = \sum_{j=1}^{h} \lambda_j E_j (E_j^T E_j)^{-1} E_j^T$$

I understand that you may feel a little overwhelmed with this formula. If we work a bit harder we can remove the presence of the annoying inverse. What I mean is that it is possible to choose a basis for each $\mathscr{R}(P_j)$ for which each of the corresponding E_j satisfy $E_j^T E_j = I$. As this construction is fairly general let us devote a separate section to it (see Gram-Schmidt Orthogonalization).

9.2: Gram-Schmidt Orthogonalization

Suppose that M is an m-dimensional subspace with basis

$$\{x_1, \cdots, x_m\}$$

We transform this into an orthonormal basis

$$\{q_1, \cdots, q_m\}$$

for M via

1. Set $y_1 = x_1$ and $q_1 = \frac{y_1}{||y_1||}$

2. $y_2 = x_2$ minus the projection of x_2 onto the line spanned by q_1.

That is,

$$y_2 = x_2 - q_1(q_1^T q_1)^{-1} q_1^T x_2 = x_2 - q_1 q_1^T x_2$$

Set $q_2 = \frac{y_2}{||y_2||}$ and $Q_2 = \{q_1, q_2\}$

3. $y_3 = x_3$ minus the projection of x_3 onto the plane spanned by q_1 and q_2. That is,

$$y_3 = x_3 - Q_2(Q_2^T Q_2)^{-1} Q_2^T x_3 = x_3 - q_1 q_1^T x_3$$

Set $q_3 = \frac{y_3}{||y_3||}$ and $Q_3 = \{q_1, q_2, q_3\}$. Continue in this fashion through step (m)

- (m) $y_m = x_m$ minus its projection onto the subspace spanned by the columns of Q_{m-1}

$$y_m = x_m - Q_{m-1}(Q_{m-1}^T Q_{m-1})^{-1} Q_{m-1}^T x_m x_m - \sum_{j=1}^{m-1} q_j q_j^T x_m$$

Set $q_m = \frac{y_m}{||y_m||}$. To take a simple example, let us orthogonalize the following basis for $\mathbb{R}^3$

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

1. $q_1 = y_1 = x_1$
2. $y_2 = x_2 - q_1 q_1^T x_2 = (\,0 \quad 1 \quad 0\,)^T$ and so, $q_2 = y_2$
3. $y_3 = x_3 - q_2 q_2^T x_3 = (\,0 \quad 0 \quad 1\,)^T$ and so, $q_3 = y_3$

We have arrived at

$$q_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad q_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad q_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Once the idea is grasped the actual calculations are best left to a machine. Matlab accomplishes this via the orth command. Its implementation is a bit more sophisticated than a blind run through our steps (1) through (m). As a result, there is no guarantee that it will return the same basis. For example

```
>>X=[1 1 1;0 1 1 ;0 0 1];

>>Q=orth(X)

Q=

  0.7370  -0.5910  0.3280
```

```
0.5910   0.3280 -0.7370

0.3280   0.7370  0.5910
```

This ambiguity does not bother us, for one orthogonal basis is as good as another. Let us put this into practice, via (10.8).

9.3: The Diagonalization of a Symmetric Matrix

By choosing an orthogonal basis $\{q_{j,k}|1 \leq k \leq n_j\}$ for each $\mathbb{R}(P_j)$ and collecting the basis vectors in

$$Q_j = \begin{pmatrix} q_{j,1} & q_{j,2} & \cdots & q_{j,n_j} \end{pmatrix}$$

We find that

$$P_j = Q_j Q_j^T = \sum_{k=1}^{n_j} q_{j,k} q_{j,k}^T$$

As a result, the spectral representation takes the form

$$B = \sum_{j=1}^{h} \lambda_j Q_j Q_j^T$$

$$\sum_{j=1}^{h} \lambda_j \sum_{k=1}^{n_j} q_{j,k} q_{j,k}^T$$

This is the spectral representation in perhaps its most detailed dress. There exists, however, still another form! It is a form that you are likely to see in future engineering courses and is achieved by assembling the Q_j into a single $n-by-n$ orthonormal matrix

$$Q = \begin{pmatrix} Q_1 & \cdots & Q_h \end{pmatrix}$$

Having orthonormal columns it follows that $Q^T Q = I$. Q being square, it follows in addition that $Q^T = Q^{-1}$. Now,

$$Bq_{j,k} = \lambda_j q_{j,k}$$

may be encoded in matrix terms via

$$BQ = Q\Lambda$$

where Λ is the $n-by-n$ diagonal matrix whose first n_1 diagonal terms are λ_1, whose next n_2 diagonal terms are λ_2, and so on. That is, each λ_j is repeated according to its multiplicity. Multiplying each side of Equation, from the right, by Q^T we arrive at

$$B = Q\Lambda Q^T$$

Because one may just as easily write

$$Q^T BQ = \Lambda$$

one says that Q **diagonalizes** B.

Let us return the our example

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

of the last chapter. Recall that the eigenspace associated with $\lambda_1 = 0$ had

$$e_{1,1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

and

$$e_{1,2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

for a basis. Via Gram-Schmidt we may replace this with

$$q_{1,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$q_{1,2} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

Normalizing the vector associated with $\lambda_2 = 3$ we arrive at

$$q_{2,1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and hence

$$Q = \begin{pmatrix} q_1^1 & q_2^1 & q_2 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{3} & -1 & \sqrt{2} \\ \sqrt{3} & -1 & \sqrt{2} \\ 0 & 2 & \sqrt{2} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

CHAPTER OVERVIEW

10: The Matrix Exponential

10.1: Overview

The matrix exponential is a powerful means for representing the solution to nn linear, constant coefficient, differential equations. The initial value problem for such a system may be written

$$x'(t) = Ax(t)$$
$$x(0) = x_0$$

where A is the n-by-n matrix of coefficients. By analogy to the 1-by-1 case we might expect

$$x(t) = e^{At}u$$

to hold. Our expectations are granted if we properly define e^{At}. Do you see why simply exponentiating each element of At will no suffice?

There are at least 4 distinct (but of course equivalent) approaches to properly defining e^{At}. The first two are natural analogs of the single variable case while the latter two make use of heavier matrix algebra machinery.

1. The Matrix Exponential as a Limit of Powers
2. The Matrix Exponential as a sum of Powers
3. The Matrix Exponential via the Laplace Transform
4. The Matrix Exponential via Eigenvalues and Eigenvectors

Please visit each of these modules to see the definition and a number of examples.

For a concrete application of these methods to a real dynamical system, please visit the Mass-Spring-Damper-module.

Regardless of the approach, the matrix exponential may be shown to obey the 3 lovely properties

1. $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$
2. $e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$
3. e^{At} is nonsingular and $(e^{At})^{-1} = e^{-(At)}$

Let us confirm each of these on the suite of examples used in the submodules.

Example 10.1.1

If

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

then

$$e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

1. $\frac{d}{dt}(e^{At}) = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$
2. $\begin{pmatrix} e^{t_1+t_2} & 0 \\ 0 & e^{2t_1+2t_2} \end{pmatrix} = \begin{pmatrix} e^{t_1}e^{t_2} & 0 \\ 0 & e^{2t_1}e^{2t_2} \end{pmatrix} = \begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{2t_1} \end{pmatrix}\begin{pmatrix} e^{t_2} & 0 \\ 0 & e^{2t_2} \end{pmatrix}$
3. $(e^{At})^{-1} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-(2t)} \end{pmatrix} = e^{-(At)}$

Example 10.1.2

If

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

then

$$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

1. $\frac{d}{dt}(e^{At}) = \begin{pmatrix} -\sin(t) & \cos(t) \\ -\cos(t) & -\sin(t) \end{pmatrix}$ and $Ae^{At} = \begin{pmatrix} -\sin(t) & \cos(t) \\ -\cos(t) & -\sin(t) \end{pmatrix}$
2. You will recognize this statement as a basic trig identity
$\begin{pmatrix} \cos(t_1+t_2) & \sin(t_1+t_2) \\ -\sin(t_1+t_2) & \cos(t_1+t_2) \end{pmatrix} = \begin{pmatrix} \cos(t_1) & \sin(t_1) \\ -\sin(t_1) & \cos(t_1) \end{pmatrix} \begin{pmatrix} \cos(t_2) & \sin(t_2) \\ -\sin(t_2) & \cos(t_2) \end{pmatrix}$
3. $(e^{At})^{-1} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} = \begin{pmatrix} \cos(-t) & -\sin(-t) \\ \sin(-t) & \cos(-t) \end{pmatrix} = e^{-(At)}$

Example 10.1.3

If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

1. $\frac{d}{dt}(e^{At}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = Ae^{At}$
2. $\begin{pmatrix} 1 & t_1+t_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix}$
3. $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = e^{-At}$

10.2: The Matrix Exponential as a Limit of Powers

You may recall from Calculus that for any numbers aa and tt one may achieve e^{at} via

$$e^{at} = \lim_{k\to\infty}(1+\frac{at}{k})^k$$

The natural matrix definition is therefore

$$e^{At} = \lim_{k\to\infty}(I+\frac{At}{k})^k$$

where I is the n-by-n identity matrix.

Example 10.2.1

The easiest case is the diagonal case, e.g.,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

for then

$$(I+\frac{At}{k})^k = \begin{pmatrix} (1+\frac{t}{k})^k & 0 \\ 0 & (1+\frac{2t}{k})^k \end{pmatrix}$$

and so

$$e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

Note that this is **NOT** the exponential of each element of A.

Example 10.2.2

As a concrete example let us suppose

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

From

$$I+At = \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}$$

$$(I+\frac{At}{2})^2 = \begin{pmatrix} 1 & \frac{t}{2} \\ \frac{-t}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{t}{2} \\ \frac{-t}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{t^2}{4} & t \\ -t & 1-\frac{t^2}{4} \end{pmatrix}$$

$$(I+\frac{At}{2})^3 = \begin{pmatrix} 1-\frac{t^2}{3} & t-\frac{t^3}{27} \\ -t+\frac{t^3}{27} & 1-\frac{t^2}{3} \end{pmatrix}$$

$$(I+\frac{At}{2})^4 = \begin{pmatrix} -\frac{3t^2}{8}+\frac{t^4}{256}+1 & t-\frac{t^3}{16} \\ -t+\frac{t^3}{16} & -\frac{3t^2}{8}+\frac{t^4}{256}+1 \end{pmatrix}$$

$$(I+\frac{At}{2})^5 = \begin{pmatrix} -\frac{2t^2}{5}+\frac{t^4}{125}+1 & t-\frac{2t^3}{25}+\frac{t^5}{3125} \\ -t+\frac{2t^3}{25}-\frac{t^5}{3125} & -\frac{2t^2}{5}+\frac{t^4}{125}+1 \end{pmatrix}$$

We discern a pattern: the diagonal elements are equal even polynomials while the off diagonal elements are equal but opposite odd polynomials. The degree of the polynomial will grow with kk and in the limit we 'recognize'

$$e^{At} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

Example 10.2.3

If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then

$$(I + \frac{At}{k})^k = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

for each value of k and so

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

10.3: The Matrix Exponential as a Sum of Powers

You may recall from Calculus that for any numbers aa and tt one may achieve eateat via

$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}$$

The natural matrix definition is therefore

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

where $A^0 = I$ is the n-by-n identity matrix.

Example 10.3.1

The easiest case is the diagonal case, e.g.,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

for then

$$(At)^k = \begin{pmatrix} t^k & 0 \\ 0 & (2t)^k \end{pmatrix}$$

and so

$$e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

Note that this is **NOT** the exponential of each element of A.

Example 10.3.2

As a second example let us suppose

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We recognize that its powers cycle, i.e.,

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = A$$

and so

$$e^{At} = \begin{pmatrix} 1 - \frac{t^2}{2} + \frac{t^4}{4} + \cdots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \cdots & 1 - \frac{t^2}{2} + \frac{t^4}{4} + \cdots \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

Example 10.3.3

If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then

$$A^2 = A^3 = A^k = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and so

$$e^{At} = (I + tA) \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

10.4: The Matrix Exponential via the Laplace Transform

You may recall from the Laplace Transform module that may achieve e^{at} via

$$e^{at} = \mathscr{L}^{-1}(\frac{1}{s-a})$$

The natural matrix definition is therefore

$$e^{At} = \mathscr{L}^{-1}((sI - A)^{-1})$$

where I is the n-by-n identity matrix.

Example 10.4.1

The easiest case is the diagonal case, e.g.,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

for then

$$(sI - A)^{-1} = \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix}$$

and so

$$e^{At} = \begin{pmatrix} \mathscr{L}^{-1}(\frac{1}{s-1}) & 0 \\ 0 & \mathscr{L}^{-1}(\frac{1}{s-2}) \end{pmatrix}$$

Example 10.4.2

As a second example let us suppose

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and compute, in matlab,

```
>> inv(s*eye(2)-A)

           ans = [ s/(s^2+1),  1/(s^2+1)]
                 [-1/(s^2+1),  s/(s^2+1)]

        >> ilaplace(ans)

           ans = [ cos(t),  sin(t)]
                 [-sin(t),  cos(t)]
```

Example 10.4.3

If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then

```
>> inv(s*eye(2)-A)

           ans = [ 1/s,  1/s^2]
                 [   0,    1/s]

        >> ilaplace(ans)

           ans = [ 1,  t]
                 [ 0,  1]
```

10.5: The Matrix Exponential via Eigenvalues and Eigenvectors

In this module we exploit the fact that the matrix exponential of a diagonal matrix is the diagonal matrix of element exponentials. In order to exploit it we need to recall that all matrices are almost diagonalizable. Let us begin with the clean case: if A is n-by-n and has n distinct eigenvalues, λ_j, and therefore n linear eigenvectors, s_j, then we note that

$$\forall j, j \in \{1, \cdots, n\} : (As_j = \lambda_j s_j)$$

maybe written as

$$A = S\Lambda S^{-1}$$

where $S = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \end{pmatrix}$ is the full matrix of eigenvectors and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n$ is the diagonal matrix of eigenvalues. One cool reason for writing A as in Equation is that

$$A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

and, more generally

$$A^k = S\Lambda^k S^{-1}$$

If we now plug this into the definition in The Matrix Exponential as a Sum of Powers, we find

$$e^{At} = Se^{\Lambda t} S^{-1}$$

where $e^{\Lambda t}$ is simply

$$diag(e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_1 t})$$

Let us exercise this on our standard suite of examples.

Example 10.5.1

If

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

then

$$S = I\Lambda = A$$

and so $e^{At} = e^{\Lambda t}$

Example 10.5.2

As a second example let us suppose

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and compute, in matlab,

```
>> [S, Lam] = eig(A)

           S = 0.7071                  0.7071
                    0 + 0.7071i               0 - 0.7071i


           Lam = 0 + 1.0000i     0
```

```
                0                0 - 1.0000i


      >> Si = inv(S)

         Si = 0.7071     0 - 0.7071i
              0.7071     0 + 0.7071i


      >> simple(S*diag(exp(diag(Lam)*t))*Si)

         ans = [ cos(t),   sin(t)]
               [-sin(t),   cos(t)]
```

Example 10.5.3

If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then matlab delivers

```
>> [S, Lam] = eig(A)

         S = 1.0000   -1.0000
             0         0.0000

         Lam = 0    0
               0    0
```

So zero is a double eigenvalue with but one eigenvector. Hence SS is not invertible and we can not invoke. The generalization is often called the Jordan Canonical Form or the Spectral Representation. The latter reads

$$A = \sum_{j=1}^{h} \lambda_j P_j + D_j$$

where the λ_j are the distinct eigenvalues of A while, in terms of the resolvent $R(z) = (zI - A)^{-1}$

$$P_j = \frac{1}{2\pi i} \int R(z) dz$$

is the associated eigen-projection and

$$D_j = \frac{1}{2\pi i} \int R(z)(z - \lambda_j) dz$$

is the associated eigen-nilpotent. In each case, C_j is a small circle enclosing only λ_j

Conversely we express the resolvent

$$R(z) = \sum_{j=1}^{h} \frac{1}{z - \lambda_j} P_j + \sum_{k=1}^{m_j - 1} \frac{1}{(z - \lambda_j)^{k+1}} D_j^k$$

where

$$m_j = \dim(\mathscr{R}(P_j))$$

with this preparation we recall Cauchy's integral formula for a smooth function f

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}dz$$

where $C(a)$ is a curve enclosing the point a

$$f(A) = \frac{-1}{2\pi i}\int f(z)R(z)dz$$

where $C(r)$ encloses **ALL** of the eigenvalues of A. For $f(z) = e^{zt}$ we find

$$e^{At} = \sum_{j=1}^{h} e^{\lambda_j t}(P_j + \sum_{k=1}^{m_j-1}\frac{t^k}{k!}D_j^k)$$

with regard to our example we find, $h = 1, \lambda_1 = 0, P_1 = I, m_1 = 2, D_1 = A$ so

$$e^{At} = I + tA$$

Let us consider a slightly bigger example, if

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

then

```
>> R = inv(s*eye(3)-A)

           R = [ 1/(s-1),   1/(s-1)^2,         0]
               [       0,     1/(s-1),         0]
               [       0,           0,   1/(s-2)]
```

and so $\lambda_1 = 1$ and $\lambda_2 = 2$ while

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so $m_1 = 2$

$$D_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $m_2 = 1$ and $D_2 = 0$

$$e^{At} = e^t(P_1 + tD_1) + e^{2t}P_2$$

$$\begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}$$

10.6: The Mass-Spring-Damper System

Figure 1. Mass, spring, damper system

If one provides an initial displacement, x_0, and velocity, v_0, to the mass depicted in Figure then one finds that its displacement, $x(t)$ at time t satisfies

$$m\frac{d^2x(t)}{dt^2}+2c\frac{dx(t)}{dt}+kx(t)=0$$

$$x(0)=x_0$$

$$x\,'(0)=v_0$$

where prime denotes differentiation with respect to time. It is customary to write this single second order equation as a pair of first order equations. More precisely, we set

$$u_1(t)=x(t)$$

$$u_2(t)=x\,'(t)$$

and note that Equation becomes

$$mu_2\,'(t)=(-(ku_1(t)))-2cu_2(t)$$

$$u_1\,'(t)=u_2(t)$$

Denoting $u(t)\equiv\begin{pmatrix}u_1(t) & u_2(t)\end{pmatrix}^T$ we write Equation as

$$\forall A,A=\begin{pmatrix}0 & 1\\ \frac{-k}{m} & \frac{-2c}{m}\end{pmatrix}:(u\,'(t)=Au(t))$$

We recall from The Matrix Exponential module that

$$u(t)=e^{At}u(0)$$

We shall proceed to compute the matrix exponential along the lines of The matrix Exponential via Eigenvalues and Eigenvectors module. To begin we record the resolvent

$$R(z)=\frac{-1}{mz^2+2cz+k}\begin{pmatrix}2c+mz & m\\ -k & mz\end{pmatrix}$$

The eigenvalues are the roots of $mz^2+2cz+k$

$$\forall d,d=\sqrt{c^2-mk}:(\lambda_1=\frac{(-c)-d}{m}$$

$$\forall d,d=\sqrt{c^2-mk}:(\lambda_2=\frac{(-c)+d}{m}$$

We naturally consider two cases, the first being

- $d\neq 0$. In this case the partial fraction expansion of $R(z)$ yields

$$R(z)=\frac{-1}{z-\lambda_1}\frac{1}{2d}\begin{pmatrix}d-c & -m\\ k & c+d\end{pmatrix}+\frac{-1}{z-\lambda_2}\frac{1}{2d}\begin{pmatrix}c+d & m\\ -k & d-c\end{pmatrix}=\frac{-1}{z-\lambda_1}P_1+\frac{-1}{z-\lambda_2}P_2$$

and so $e^{At}=e^{\lambda_1 t}P_1+e^{\lambda_2 t}P_2$ i.e., v_0 it follows that

$$x(t)=\frac{x_0}{2d}(e^{\lambda_1 t}(d-c)+e^{\lambda_2 t}(c+d))$$

If d is real, i.e., if $c^2>mk$ then both λ_1 and λ_2 are negative real numbers and $x(t)$ decays to 0 without oscillation. If, on the contrary, d is imaginary, i.e., $c^2<mk$, then

$$x(t) = e^{-(ct)}(\cos(|d|t) + \frac{c}{|d|}\sin(|d|t))$$

and so x decays to 0 in an oscillatory fashion. When Equation holds the system is said to be **overdamped** while when Equation governs then we speak of the system as **underdamped**. It remains to discuss the case of **critical damping**.

- $d = 0$. In this case, $\lambda_1 = \lambda_2 = -\sqrt{\frac{k}{m}}$ and so we need only compute P_1 and D_1. As there is but one P_j and the P_j are known to sum to the identity it follows that $P_1 = I$. Similarly, this equation dictates that

$$D_1 = AP_1 - \lambda_1 P_1 = A - \lambda_1 I = \begin{pmatrix} \sqrt{\frac{k}{m}} & 1 \\ -\frac{k}{m} & \sqrt{\frac{k}{m}} \end{pmatrix}$$

On substitution of this into this equation we find

$$e^{At} = e^{-(t\sqrt{\frac{k}{m}})} \begin{pmatrix} 1 + t\sqrt{\frac{k}{m}} & 1 \\ -(t\frac{k}{m}) & 1 - t\sqrt{\frac{k}{m}} \end{pmatrix}$$

Under the assumption, as above, that $v_0 = 0$, we deduce from Equation that

$$x(t) = e^{-(t\sqrt{\frac{k}{m}})}(1 + t\sqrt{\frac{k}{m}})x_0$$

CHAPTER OVERVIEW

11: Singular Value Decomposition

11.1: The Singular Value Decomposition

,The singular value decomposition is another name for the spectral representation of a rectangular matrix. Of course if A is m-by-m and $m \neq n$ then it does not make sense to speak of the eigenvalues of A. We may, however, rely on the previous section to give us relevant spectral representations of the two symmetric matrices

- $A^T A$
- AA^T

That these two matrices together may indeed tell us 'everything' about A can be gleaned from

$$\mathcal{N}(A^T A) = \mathcal{N}(A)$$

$$\mathcal{N}(AA^T) = \mathcal{N}(A^T)$$

$$\mathcal{R}(A^T A) = \mathcal{R}(A^T)$$

$$\mathcal{R}(AA^T) = \mathcal{R}(A)$$

You have proven the first of these in a previous exercise. The proof of the second is identical. The row and column space results follow from the first two via orthogonality.

On the spectral side, we shall now see that the eigenvalues of AA^T and $A^T A$ are nonnegative and that their nonzero eigenvalues coincide. Let us first confirm this on the adjacency matrix associated with the unstable swing

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The respective products are

$$AA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Analysis of the first is particularly simple. Its null space is clearly just the zero vector while $\lambda_1 = 2$ and $\lambda_2 = 1$ are its eigenvalues. Their geometric multiplicities are $n_1 = 1$ and $n_2 = 2$. In $A^T A$ we recognize the S matrix from the exercise in another module and recall that its eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = 0$ with multiplicities $n_1 = 1$, $n_2 = 2$, and $n_3 = 1$. Hence, at least for this A, the eigenvalues of AA^T and $A^T A$ are nonnegative and their nonzero eigenvalues coincide. In addition, the geometric multiplicities of the nonzero eigenvalues sum to 3, the rank of A.

Proposition

The eigenvalues of AA^T and $A^T A$ are nonnegative. Their nonzero eigenvalues, including geometric multiplicities, coincide. The geometric multiplicities of the nonzero eigenvalues sum to the rank of A.

If $A^T Ax = \lambda x$ then $x^T A^T Ax = \lambda x^T x$, i.e., $(||Ax||)^2 = \lambda(||x||)^2$ and so $\lambda \geq 0$. A similar argument works for AA^T.

Now suppose that $\lambda_j > 0$ and that $\{x_{j,k}\}_{k=1}^{n_j}$ constitutes an orthogonal basis for the eigenspace $\mathcal{R}(P_j)$, starting from

$$A^T Ax_{j,k} = \lambda_j x_{j,k}$$

we find, on multiplying through (from the left) by A that

$$AA^T Ax_{j,k} = \lambda_j Ax_{j,k}$$

i.e., λ_j is an eigenvalue of AA^T with eigenvector $Ax_{j,k}$, so long as $Ax_{j,k} \neq 0$.

It follows from the first paragraph of this proof that $||Ax_{j,k}|| = \sqrt{\lambda_j}$, which, by hypothesis, is nonzero. Hence,

$$\forall 1 \leq k \leq n_j : (y_{j,k} \equiv \frac{Ax_{j,k}}{\sqrt{\lambda_j}}$$

is a collection of unit eigenvectors of AA^T associated with λ_j. Let us now show that these vectors are orthonormal for fixed j.

$$y_{j,i}^T y_{j,k} = \frac{1}{\lambda_j} x_{j,i}^T A^T A x_{j,k} = x_{j,i}^T x_{j,k} = 0$$

We have now demonstrated that if $\lambda_j > 0$ is an eigenvalue of $A^T A$ of geometric multiplicity n_j. Reversing the argument, i.e., generating eigenvectors of $A^T A$ from those of AA^T we find that the geometric multiplicities must indeed coincide.

Regarding the rank statement, we discern from Equation that if $\lambda_j > 0$ then $x_{j,k} \in \mathscr{R}(A^T A)$. The union of these vectors indeed constitutes a basis for $\mathscr{R}(A^T A)$, for anything orthogonal to each of these $x_{j,k}$ necessarily lies in the eigenspace corresponding to a zero eigenvalue, i.e., in $\mathscr{N}(A^T A)$. As $\mathscr{R}(A^T A) = \mathscr{R}(A^T)$ it follows that $\dim \mathscr{R}(A^T A) = r \dim \mathscr{R} A^T A = r$ and hence the n_j, for $\lambda_j > 0$, sum to r.

Let us now gather together some of the separate pieces of the proof. For starters, we order the eigenvalues of $A^T A$ from high to low,

$$\lambda_1 > \lambda_2 > \cdots > \lambda_h$$

and write

$$A^T A = X \Lambda_n X^T$$

where

$$\forall X_j = \{x_{j,1}, \cdots, x_{j,n_j}\} : (X = \{X_1, \cdots, X_h\})$$

and Λ_n is the $n - by - n$ diagonal matrix with λ_1 in the first n_1 slots, λ_2 in the next n_2 slots, etc. Similarly

$$AA^T = Y \Lambda_m Y^T$$

where

$$\forall Y_j = \{y_{j,1}, \cdots, y_{j,n_j}\} : (Y = \{Y_1, \cdots, Y_h\})$$

and Λ_m is the mmmm diagonal matrix with λ_1 in the first n_1 slots, λ_2 in the next n_2 slots, etc. The $y_{j,k}$ were defined in Equation under the assumption that $\lambda_j > 0$. If $\lambda_j = 0$ let Y_j denote an orthonormal basis for $\mathscr{N}(AA^T)$. Finally, call

$$\sigma_j = \sqrt{\lambda_j}$$

and let Σ denote the m-by-n matrix diagonal matrix with σ_1 in the first n_1 slots, σ_2 in the next n_2 slots, etc. Notice that

$$\Sigma^T \Sigma = \Lambda_n$$

$$\Sigma \Sigma^T = \Lambda_m$$

Now recognize that Equation may be written

$$Ax_{j,k} = \sigma_j y_{j,k}$$

and that this is simply the column by column rendition of

$$AX = Y\Sigma$$

As $XX^T = I$ we may multiply through (from the right) by X^T and arrive at the **singular value decomposition** of A

$$A = Y \Sigma X^T$$

Let us confirm this on the A matrix in Equation. We have

$$\Lambda_4 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\Lambda_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence

$$\Lambda = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and so $A = Y\Sigma X^T$ says that A should coincide with

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{11.1.1}$$

This indeed agrees with A. It also agrees (up to sign changes on the columns of X with what one receives upon typing `[Y, SIG, X] = scd(A)` in Matlab.

You now ask what we get for our troubles. I express the first dividend as a proposition that looks to me like a quantitative version of the fundamental theorem of linear algebra.

Proposition

If $Y\Sigma X^T$ is the singular value decomposition of A then

1. The rank of A, call it r, is the number of nonzero elements in Σ
2. The first r columns of X constitute an orthonormal basis for $\mathscr{R}(A^T)$. The $n-r$ last columns of X constitute an orthonormal basis for $\mathscr{N}(A)$
3. The first r columns of Y constitute an orthonormal basis for $\mathscr{R}(A)$. The $m-r$ last columns of Y constitute an orthonormal basis for $\mathscr{N}(A^T)$

Let us now 'solve' $A\mathbf{x} = \mathbf{b}$ with the help of the pseudo-inverse of A. You know the 'right' thing to do, namely reciprocate all of the nonzero singular values. Because m is not necessarily n we must also be careful with dimensions. To be precise, let Σ^+ denote the $n-by-m$ matrix whose first n_1 diagonal elements are $\frac{1}{\sigma_1}$, whose next n_2 diagonal elements are $\frac{1}{\sigma_2}$ and so on. In the case that $\sigma_h = 0$, set the final n_h diagonal elements of Σ^+ to zero. Now, one defines the **pseudo-inverse** of A to be

$$A^+ \equiv X\Sigma^+ Y^T$$

In the case of that A is that appearing in Equation we find

$$\Sigma^{+}=\begin{pmatrix}\sqrt{2} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

and so

$$\begin{pmatrix}-\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{11.1.2}$$

therefore

$$A^{+}=\begin{pmatrix}0 & \frac{-1}{2} & 0\\ 1 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & 1\end{pmatrix}$$

in agreement with what appears from `pinv(A)` . Let us now investigate the sense in which A^{+} is the inverse of A. Suppose that $b\in\mathbb{R}^{m}$ and that we wish to solve $A\mathbf{x}=\mathbf{b}$. We suspect that $A^{+}b$ should be a good candidate. Observe by Equation that

$$(A^{T}A)A^{+}b=X\Lambda_{n}X^{T}X\Sigma^{+}Y^{T}b$$

because $X^{T}X=I$

$$(A^{T}A)A^{+}b=X\Lambda_{n}\Sigma^{+}Y^{T}b$$

$$(A^{T}A)A^{+}b=X\Sigma^{T}\Sigma\sigma^{+}Y^{T}b$$

because $\Sigma^{T}\Sigma\Sigma^{+}=\Sigma^{T}$

$$(A^{T}A)A^{+}b=X\Sigma^{T}Y^{T}b$$

$$(A^{T}A)A^{+}b=A^{T}b$$

that is $A^{+}b$ satisfies the **least-squares problem** $A^{T}Ax=A^{T}b$.

Index

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